

Various Theorems on Tournaments

Gaku Liu

Department of Mathematics

Princeton University

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Adviser: Paul Seymour

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Abstract

In this thesis we prove a variety of theorems on tournaments. A *prime* tournament is a tournament G such that there is no $X \subseteq V(G)$, $1 < |X| < |V(G)|$, such that for every vertex $v \in V(G) \setminus X$, either $v \rightarrow x$ for all $x \in X$ or $x \rightarrow v$ for all $x \in X$. First, we prove that given a prime tournament G which is not in one of three special families of tournaments, for any prime subtournament H of G with $5 \leq |V(H)| < |V(G)|$ there exists a prime subtournament of G with $|V(H)| + 1$ vertices that has a subtournament isomorphic to H . We next prove that for any two cyclic triangles C, C' in a prime tournament G , there is a sequence of cyclic triangles C_1, \dots, C_n such that $C_1 = C$, $C_n = C'$, and C_i shares an edge with C_{i+1} for all $1 \leq i \leq n - 1$. Next, we consider what we call *matching tournaments*, tournaments whose vertices can be ordered in a horizontal line so that every vertex is the head or tail of at most one edge that points right-to-left. We determine the conditions under which a tournament can have two different orderings satisfying the above conditions. We also prove that there are infinitely many minimal tournaments that are not matching tournaments. Finally, we consider the tournaments K_n and K_n^* , which are obtained from the transitive tournament with n vertices by reversing the edge from the second vertex to the last vertex and from the first vertex to the second-to-last vertex, respectively. We prove a structure theorem describing tournaments which exclude K_n and K_n^* as subtournaments.

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1 Introduction

A *tournament* G is a non-null, loopless directed graph such that for any two distinct vertices $u, v \in V(G)$, there is exactly one edge with both ends in $\{u, v\}$. A *subtournament* of a tournament G is a tournament induced on a nonempty subset of $V(G)$. If H is a subtournament of G and $X \subseteq V(G)$, we use $H + X$ to denote the subtournament of G induced on $V(H) \cup X$. If $X \subsetneq V(G)$, we use $G - X$ to denote the subtournament induced on $V(G) \setminus X$. We use $H + v$ to mean $H + \{v\}$ and $G - v$ to mean $G - \{v\}$. For each vertex $v \in V(G)$, let $A_G(v) = \{u \in V(G) : v \rightarrow u\}$ be the set of *outneighbors* of v in G , and let $B_G(v) = \{u \in V(G) : u \rightarrow v\}$ be the set of *inneighbors* of v in G . We call $|A_G(v)|$ the *outdegree* of v in G and $|B_G(v)|$ the *indegree* of v in G . For any two disjoint sets $X, Y \subseteq V(G)$, we write $X \Rightarrow Y$ if $x \rightarrow y$ for every $x \in X$ and $y \in Y$. We use $v \Rightarrow X$ to mean $\{v\} \Rightarrow X$. Given distinct vertices $u, v \in V(G)$, define $d_{uv} \in \{+, -\}$ so that $d_{uv} = +$ if $u \rightarrow v$ and $d_{uv} = -$ if $v \rightarrow u$. Finally, an *ordering* of $V(G)$ is a list v_1, \dots, v_n of the vertices of G . A *transitive* tournament is a tournament whose vertices can be ordered v_1, \dots, v_n such that $v_i \rightarrow v_j$ if $i < j$. We call the unique ordering v_1, \dots, v_n which satisfies the previous condition the *standard ordering* of the vertices of a transitive tournament. We use I_n to denote the isomorphism class of transitive tournaments with n vertices. We will often refer to I_n as a tournament itself; likewise, when we define other isomorphism classes of tournaments, we will often refer to them as tournaments themselves.

This thesis contains various results on tournaments proven over the course of the year. While the different major results are largely independent of each other, there are a few common ideas. Section 2 introduces homogeneous sets, the “substitution” construction, and prime tournaments, concepts that are central to this paper and will be used throughout. Sections 3 through 6 each showcase a different theorem. Section 3 gives a theorem that allows us to “grow” a prime tournament one vertex at a time starting from any of its prime subtournaments. This result strengthens a theorem by Schmerl and Trotter [8] which is often used in the study of prime tournaments (see for example [1], [2], [6]). In Section 4 we prove an interesting result on the structure of cyclic triangles within prime tournaments. Sections 5 and 6 deal with tournaments which are in a sense “almost-transitive”—they are formed from transitive tournaments by reversing a set of edges satisfying a given property. Section 5 deals with matching tournaments, tournaments for which this set of edges is a matching. Section 6 deals with tournaments with only a single edge reversed, and considers the structure of tournaments that exclude these. The final section considers directions for future research.

While this thesis is meant to be read chronologically, Sections 3 through 6 are more or less independent of each other with the following exceptions: the proofs of Theorems 4.1 and 6.2 use the concept of weaves defined in the proof of Theorem 3.1, and Section 6 uses the definition of a backedge given at the beginning of Section 5.

2 Homogeneous sets and prime tournaments

2.1 Basic definitions and properties

Given a tournament G , a *homogeneous set* of G is a subset of vertices $X \subseteq V(G)$ such that for all vertices $v \in V(G) \setminus X$, either $v \Rightarrow X$ or $X \Rightarrow v$. A homogeneous set $X \subseteq V(G)$ is *nontrivial* if $1 < |X| < |V(G)|$; otherwise it is *trivial*.

We list some basic properties of homogeneous sets.

Proposition 2.1 (Restriction). *If X is a homogeneous set of a tournament G , and H is a subtournament of G , then $X \cap V(H)$ is a homogeneous set of H .*

Proposition 2.2 (Extension). *If H_1, H_2 are subtournaments of a tournament G and $X \subseteq V(H_1) \cap V(H_2)$ is a homogeneous set of both H_1 and H_2 , then X is a homogeneous set of the tournament induced on $V(H_1) \cup V(H_2)$.*

Proposition 2.3 (Cloning). *Let G be a tournament, $x, y \in V(G)$ be distinct vertices, and $X \subseteq V(G) \setminus \{y\}$. If X is a homogeneous set of $G - y$ and $\{x, y\}$ is a homogeneous set of G , then X is a homogeneous set of G .*

Proposition 2.4 (Intersection). *If X, Y are homogeneous sets of a tournament G , then $X \cap Y$ is a homogeneous set of G .*

Proposition 2.5 (Subtraction). *If X, Y are homogeneous sets of a tournament G and $Y \setminus X \neq \emptyset$, then $X \setminus Y$ is a homogeneous set of G .*

Proposition 2.6 (Union). *Suppose G is a tournament and $X, Y \subseteq V(G)$ such that $X \cap Y \neq \emptyset$, X is a homogeneous set of $G - (Y \setminus X)$, and Y is a homogeneous set of $G - (X \setminus Y)$. Then $X \cup Y$ is a homogeneous set of G .*

A tournament is *prime* if all of its homogeneous sets are trivial; otherwise, it is *decomposable*. Given a tournament G and an ordering v_1, \dots, v_n of its vertices, and given tournaments H_1, \dots, H_n , let $G(H_1, \dots, H_n)$ be a tournament with vertex set $V_1 \cup V_2 \cup \dots \cup V_n$, where the V_i are pairwise disjoint sets of vertices, such that $V_i \Rightarrow V_j$ if $v_i \rightarrow v_j$, and for all $1 \leq i \leq n$ the subtournament of $G(H_1, \dots, H_n)$ induced on V_i is isomorphic to H_i . Every tournament with at least two vertices can be written as $G'(H_1, \dots, H_n)$ where G' is a prime tournament with at least two vertices. The prime tournaments are precisely those tournaments G which cannot be written as $G'(H_1, \dots, H_n)$ for some G' with $2 \leq |V(G')| < |V(G)|$.

A tournament G is *strongly connected* if for any two vertices $u, v \in V(G)$, there is a directed path from u to v and a directed path from v to u . A *strongly connected component*, or *strong component*, of a tournament G is a maximal strongly connected subtournament of G . The strong components of a tournament G can be ordered as S_1, \dots, S_s so that G can be written as $I_s(S_1, \dots, S_s)$, where I_s has the standard ordering for a transitive tournament. From this we see that the vertices of a strong component of a tournament form a homogeneous set, so if a tournament is prime,

either it is strongly connected or all of its strong components have only one vertex. In the latter case the tournament is transitive, and I_n has a homogeneous set for $n \geq 3$. So every prime tournament with ≥ 3 vertices is strongly connected.

The tournament with 1 vertex and the tournament with 2 vertices are both prime. The only prime tournament with 3 vertices is the cyclic triangle. It can be checked that all tournaments with 4 vertices are decomposable. For 5 vertices, there are exactly three prime tournaments T_5 , U_5 , and W_5 , drawn below.

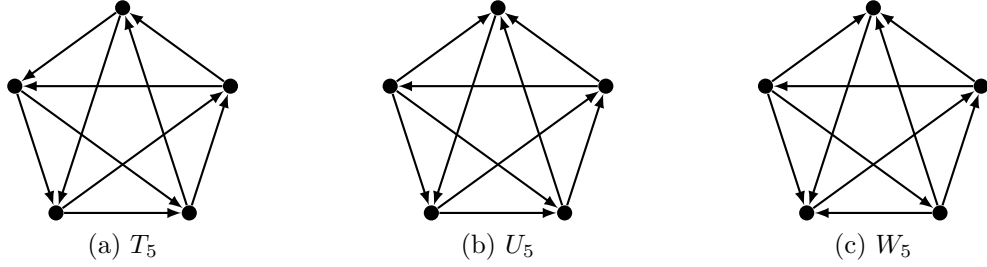


Figure 1: The three five-vertex prime tournaments

These three tournaments can be generalized to any odd number of vertices as follows.

Definition 2.7. Let $k \geq 0$ and $n = 2k + 1$. Define the tournaments T_n , U_n , and W_n as follows.

- T_n is the tournament with vertices v_1, \dots, v_n such that $v_i \rightarrow v_j$ if $j \equiv i + 1, i + 2, \dots, \text{ or } i + k \pmod{n}$.
- U_n is the tournament obtained from T_n by reversing all edges which have both ends in $\{v_1, \dots, v_k\}$.
- W_n is the tournament with vertices w_1, \dots, w_n such that $w_i \rightarrow w_j$ if $1 \leq i < j \leq n - 1$, and $\{w_2, w_4, \dots, w_{n-1}\} \Rightarrow w_n \Rightarrow \{w_1, w_3, \dots, w_{n-2}\}$.

Note the following degenerate cases: T_1 , U_1 , and W_1 are all the single-vertex tournament, and T_3 , U_3 , and W_3 are all the cyclic triangle.

It can be checked that T_n , U_n , and W_n are prime for all odd n . Also, note that T_n , U_n , and W_n have subtournaments isomorphic to T_m , U_m , and W_m , respectively, for $m \leq n$. In fact, the only prime subtournaments of T_n , U_n , and W_n with at least 3 vertices are T_m , U_m , and W_m , respectively, for $3 \leq m \leq n$.

2.2 Prime subtournaments of prime tournaments

Given a prime tournament, it is natural to ask about its prime subtournaments. There have been several results proven about prime subtournaments; to state them, we introduce some extra terminology. Let G be a tournament and H be a prime subtournament of G with $|V(H)| \geq 3$. Define $\text{Ext}(H)$ to be the set of vertices

$v \in V(G) \setminus V(H)$ for which the tournament induced on $H + v$ is prime. Let $Z(H)$ be the set of vertices $v \in V(G) \setminus V(H)$ for which $V(H)$ is a homogeneous set of $H + v$. Finally, for each $x \in V(H)$, let $V_x(H)$ be the set of vertices $v \in V(G) \setminus V(H)$ for which $\{v, x\}$ is a homogeneous set of $H + v$.

We can use the basic properties of homogeneous tournaments found in Propositions 2.1 through 2.6 to prove the following, which appears in [4].

Proposition 2.8 (Ehrenfeucht and Rozenberg [4]). *Let G be a tournament and H be a prime subtournament of G with $|V(H)| \geq 3$. Then*

- (i) *The collection of sets $\{\text{Ext}(H), Z(H)\} \cup \{V_x(H) : x \in V(H)\}$ forms a partition of $V(G) \setminus V(H)$.*
- (ii) *If $u, v \in \text{Ext}(H)$, $u \neq v$, such that $H + \{u, v\}$ is decomposable, then $\{u, v\}$ is a homogeneous set of $H + \{u, v\}$.*
- (iii) *If $u \in Z(H)$ and $v \in (V(G) \setminus V(H)) \setminus Z(H)$ such that $H + \{u, v\}$ is decomposable, then $V(H) \cup \{v\}$ is a homogeneous set of $H + \{u, v\}$.*
- (iv) *If $u \in V_x(H)$ for some $x \in V(H)$ and $v \in (V(G) \setminus V(H)) \setminus V_x(H)$ such that $H + \{u, v\}$ is decomposable, then $\{u, x\}$ is a homogeneous set of $H + \{u, v\}$.*

Proof. We will only prove part (i); the proofs of the other parts involve similar techniques. We need to show that if $v \in V(G) \setminus V(H)$ is a vertex such that $H + v$ is decomposable, then v is in exactly one of the sets $Z(H)$, $V_x(H)$ for $x \in V(H)$. First, we show that v is in at least one of these sets. Since $H + v$ is decomposable, it has a nontrivial homogeneous set $X \subseteq V(H + v)$. By restriction to H , we have that $X \cap V(H)$ is a homogeneous set of H . Since H is prime, we must thus have either $|X \cap V(H)| \leq 1$ or $X \cap V(H) = V(H)$. If the former case, then since X is a nontrivial homogeneous set of $H + v$, we must have $|X \cap V(H)| = 1$ and $v \in X$, so $v \in V_{X \cap V(H)}(H)$, as desired. If the latter case, then $v \in Z(H)$, as desired. So v is in at least one of $Z(H)$, $V_x(H)$ for $x \in V(H)$.

Suppose v is in at least two of these sets. First, suppose $v \in Z(H)$ and $v \in V_x(H)$ where $x \in V(H)$. We have that $V(H)$ and $\{v, x\}$ are homogeneous sets of $H + v$. Applying Proposition 2.5, we have that $V(H) \setminus \{x\}$ is a homogeneous set of $H + v$, and restricting to H we have that $V(H) \setminus \{x\}$ is a homogeneous set of H . Since $|V(H)| \geq 3$, this is a nontrivial homogeneous set of H , a contradiction.

Now suppose $v \in V_x(H)$ and $v \in V_y(H)$ for two distinct $x, y \in V(H)$. Then $\{v, x\}$ and $\{v, y\}$ are homogeneous sets of $H + v$. Applying Proposition 2.6, we have that $\{v, x, y\}$ is a homogeneous set of $H + v$, and restricting to H , we have that $\{x, y\}$ is a homogeneous set of H . Since $|V(H)| \geq 3$, this is a nontrivial homogeneous set of H , a contradiction. This completes the proof. \square

This result leads to the following corollaries in the case where G is prime.

Corollary 2.9 (Ehrenfeucht and Rozenberg [4]). *Let G be a prime tournament and H be a prime subtournament of G with $3 \leq |V(H)| \leq |V(G)| - 2$. Then there exist distinct vertices $u, v \in V(G) \setminus V(H)$ such that $H + \{u, v\}$ is prime.*

Proof. Suppose that for every distinct $u, v \in V(G) \setminus V(H)$, $H + \{u, v\}$ is decomposable. We first prove that $Z(H)$ and $V_x(H)$ are empty for all $x \in V(H)$. By Proposition 2.8(iii), we have that $V(H) \cup \{v\}$ is a homogeneous set of $H + \{u, v\}$ for every $u \in Z(H)$ and $v \in (V(G) \setminus V(H)) \setminus Z(H)$. Repeatedly applying Propositions 2.2 and 2.6 on these homogeneous sets, we thus have that $V(G) \setminus Z(H)$ is a homogeneous set of G . If $Z(H)$ is nonempty, then $V(G) \setminus Z(H)$ is a nontrivial homogeneous set of G , contradicting the fact that G is prime. So $Z(H)$ is empty.

Similarly, for each $x \in V(H)$, we have by Proposition 2.8(iv) that $\{u, x\}$ is a homogeneous set of $H + \{u, v\}$ for every $u \in V_x(H)$, $v \in (V(G) \setminus V(H)) \setminus V_x(H)$. Repeatedly applying Propositions 2.2 and 2.6 on these homogeneous sets, we have that $V_x(H) \cup \{x\}$ is a homogeneous set of G . If $V_x(H)$ is nonempty, then $V_x(H) \cup \{x\}$ is a nontrivial homogeneous set of G , a contradiction. So $V_x(H)$ is empty for each $x \in V(H)$.

It follows by Proposition 2.8(i) that $V(G) \setminus V(H) = \text{Ext}(H)$. Thus, by Proposition 2.8(ii), $\{u, v\}$ is a homogeneous set of $H + \{u, v\}$ for every distinct $u, v \in V(G) \setminus V(H)$. Since $|V(G) \setminus V(H)| \geq 2$, we can apply Proposition 2.6 repeatedly on these homogeneous sets to get that $V(G) \setminus V(H)$ is a homogeneous set of G . This is a nontrivial homogeneous set of G , a contradiction. \square

Corollary 2.10. *Every prime tournament G with $|V(G)| \geq 5$ has a prime subtournament with 5 vertices.*

Proof. Since G is prime and has > 2 vertices, it is not transitive, so it contains a cyclic triangle. Applying Corollary 2.9 with H as a cyclic triangle gives the corollary. \square

Corollary 2.9 can be thought of as a “growing” lemma: within a prime tournament, we can grow an increasing sequence of prime subtournaments so that each subtournament contains the previous one. For example, starting with a cyclic triangle in a prime tournament G and repeatedly applying the corollary, we have that G contains a prime subtournament with n vertices for every odd $n < |V(G)|$. In particular, G contains a prime subtournament with either $|V(G)| - 2$ or $|V(G)| - 1$ vertices. This last statement was improved upon by Schmerl and Trotter in [8].

Theorem 2.11 (Schmerl and Trotter [8]). *If G is a prime tournament with $|V(G)| \geq 6$, and G is not T_n , U_n , or W_n for any odd n , then G has a prime subtournament with $|V(G)| - 1$ vertices.*

Theorem 2.12 (Schmerl and Trotter [8]). *If G is a prime tournament with $|V(G)| \geq 7$, then G has a prime subtournament with $|V(G)| - 2$ vertices.*

Schmerl and Trotter’s proof of these two theorems involve “growing” prime subtournaments, during which Corollary 2.9 is essential. Our result in the next section

can be thought of as a strengthening of Corollary 2.9 in that it allows us to grow the sequence of prime subtournaments one vertex at a time instead of two vertices at a time in the case where G is not T_n , U_n , or W_n . In particular, our theorem has both Theorems 2.11 and 2.12 as immediate corollaries.

3 Growing prime tournaments

3.1 Statement of theorem

The main theorem is as follows.

Theorem 3.1. *Let G be a prime tournament which is not T_n , U_n , or W_n for any odd n , and let H be a prime subtournament of G with $5 \leq |V(H)| \leq |V(G)| - 1$. Then there exists a prime subtournament of G with $|V(H)| + 1$ vertices that has a subtournament isomorphic to H .*

Note that Theorem 3.1 is not a strict strengthening of Corollary 2.9 because the theorem does not guarantee that the subtournament with $|V(H)| + 1$ vertices contains the actual vertices of H ; it only guarantees that it contains a subtournament isomorphic to H . However, in many applications where one would want to grow prime subtournaments, only the isomorphism class of the previous subtournament matters; for example, to use this theorem to prove Theorems 2.11 and 2.12, only the number of vertices at each step matters.

Theorem 3.1 is based on and is a direct analogue of a theorem by Chudnovsky and Seymour for undirected graphs, found in [3]. Indeed, the proof for Theorem 3.1 found here is closely related to the proof found in [3]. Chudnovsky and Seymour used their theorem to develop a polynomial-time algorithm to find simplicial cliques in prime claw-free graphs.

3.2 Proof of theorem

Our proof consists of two main claims.

Claim 1. *The theorem holds when $|V(H)| \geq |V(G)| - 2$.*

Claim 2. *If $n \geq 7$ is odd and G is a prime tournament with $n + 1$ vertices that has a subtournament isomorphic to T_n (U_n , W_n , respectively), then G has a prime subtournament with $n - 1$ vertices that has a subtournament isomorphic to T_{n-2} (U_{n-2} , W_{n-2} , respectively).*

Assuming the truth of these two claims, we can prove Theorem 3.1 as follows: By Claim 1, we can assume $|V(H)| < |V(G)| - 2$. First suppose that H is not T_n , U_n , or W_n for any n . By Corollary 2.9, there are vertices $\{u, v\} \in V(G) \setminus V(H)$ such that $H + \{u, v\}$ is prime. Since H is not T_n , U_n , or W_n for any n , $H + \{u, v\}$ is not any of these tournaments either (see the second paragraph after Definition 2.7). So

applying Claim 1 to $H + \{u, v\}$ with subtournament H , we have a subtournament of $H + \{u, v\}$ with $|V(H)| + 1$ vertices that has a subtournament isomorphic to H , as desired.

Now assume H is T_n , U_n , or W_n for some odd $n \geq 5$. We will assume H is T_n ; the arguments for U_n and W_n are identical. Suppose there is no prime subtournament of G with $n + 1$ vertices that has a subtournament isomorphic to T_n . Let m be the largest odd integer such that

- G has a subtournament isomorphic to T_m , and
- G has no prime subtournament with $m + 1$ vertices that has a subtournament isomorphic to T_m .

Thus, $m \geq n \geq 5$. By Claim 1, we have $m < |V(G)| - 2$. We claim that G has a subtournament isomorphic to T_{m+2} . Applying Corollary 2.9 on a copy of T_m in G , we have that G has a prime subtournament H_{m+2} with $m + 2$ vertices that has a subtournament isomorphic to T_m . If H_{m+2} is not T_{m+2} , then we can apply Claim 1 to it to obtain a prime subtournament with $m + 1$ vertices that has a subtournament isomorphic to T_m , contradicting the definition of m . Thus, H_{m+2} is T_{m+2} .

Thus, G has a subtournament isomorphic to T_{m+2} . Hence, by the maximality of m , there is a prime subtournament H_{m+3} of G with $m + 3$ vertices that has a subtournament isomorphic to T_{m+2} . Applying Claim 2, H_{m+3} has a prime subtournament with $m + 1$ vertices that has a subtournament isomorphic to T_m . This contradicts the definition of m , completing the proof.

We now prove the two claims.

Proof of Claim 1. The proof of this claim is made considerably simpler by establishing the right definitions, so we will spend a good amount of time doing so. These definitions will also be useful in later proofs.

We define a *weave* $\langle w_1, \dots, w_n \rangle$ to be a tournament with vertices w_1, \dots, w_n such that

- (1) $w_i \rightarrow w_j$ if $i < j$ and i, j have opposite parity
- (2) One of the following holds:
 - (2a) $w_i \rightarrow w_j$ for all $i < j$ with i, j odd
 - (2b) $w_j \rightarrow w_i$ for all $i < j$ with i, j odd
- (3) One of the following holds:
 - (3a) $w_i \rightarrow w_j$ for all $i < j$ with i, j even
 - (3b) $w_j \rightarrow w_i$ for all $i < j$ with i, j even

Using “F” to mean “forward” and “B” to mean “backward,” we will call a weave an FF weave if (2a) and (3a) hold, an FB weave if (2a) and (3b) hold, a BF weave if (2b) and (3a) hold, and a BB weave if (2b) and (3b) hold. We refer to these as the four *types* of weaves. (A weave can be of more than one type if $n \leq 3$.)

The following facts can be easily checked; the information that is most relevant to our proof is summarized in the corollary afterwards.

Proposition 3.2. *Let $W = \langle w_1, \dots, w_n \rangle$ be a weave.*

- *If W is FF, then it is transitive.*
- *If n is even and W is FB, then $\{w_2, \dots, w_n\}$ is a homogeneous set of W .*
- *If n is even and W is BF, then $\{w_1, \dots, w_{n-1}\}$ is a homogeneous set of W .*
- *If n is even and W is BB, then $\{w_1, w_n\}$ is a homogeneous set of W .*
- *If n is odd and W is FB, then $\{w_1, \dots, w_{n-1}\}$ is a homogeneous set of W .*
- *If n is odd and W is BF, then W is U_n .*
- *If n is odd and W is BB, then W is T_n .*

Let $v \notin V(W)$ be a vertex such that $v \rightarrow w_i$ for odd i and $w_i \rightarrow v$ for even i .

- *If n is odd and W is FF or FB, then $\{v, w_1, \dots, w_{n-1}\}$ is a homogeneous set of $W + v$.*
- *If n is odd and W is BF or BB, then $\{v, w_n\}$ is a homogeneous set of $W + v$.*
- *If n is even and W is FF, then $W + v$ is W_{n+1} .*
- *If n is even and W is FB or BF, then $W + v$ is U_{n+1} .*
- *If n is even and W is BB, then $W + v$ is T_{n+1} .*

Corollary 3.3. *Let W be a weave.*

- (i) *If $|V(W)| \geq 3$ and W is prime, then W is T_n or U_n for some n .*
- (ii) *If $|V(W)| \geq 2$ and $v \notin V(W)$ is a vertex as in Proposition 3.2, and $W + v$ is prime, then $W + v$ is T_n , U_n , or W_n for some n .*

The following fact will also be important.

Proposition 3.4. *If $W = \langle w_1, \dots, w_n \rangle$ is a weave and $1 \leq i \leq n - 1$, then $W - \{w_i, w_{i+1}\}$ is a weave of the same type as W .*

Corollary 3.5. *If $W = \langle w_1, \dots, w_n \rangle$ is a weave, then the subtournaments $W - \{w_1, w_2\}$, $W - \{w_2, w_3\}$, \dots , $W - \{w_{n-1}, w_n\}$ are all isomorphic to each other. Furthermore, there is an isomorphism ϕ between $W - \{w_i, w_{i+1}\}$ and $W - \{w_j, w_{j+1}\}$ such that if $\phi(w_k) = w_\ell$, then k and ℓ have the same parity.*

We are now ready to prove the claim. The case $|V(H)| = |V(G)| - 1$ is trivial, so assume $|V(H)| = |V(G)| - 2$. Suppose there is no prime subtournament of G with $|V(H)| + 1$ vertices that has a subtournament isomorphic to H . Let $\{u, v\} = V(G) \setminus V(H)$, where $u \rightarrow v$. We will call a weave $W = \langle w_1, \dots, w_n \rangle$ a u, v -weave if all of the following hold:

- (a) W is a subtournament of G .
- (b) $u = w_j$ and $v = w_{j+1}$ for some $1 \leq j \leq n - 1$.
- (c) $\{w_1, w_3, \dots\}$ is a homogeneous set in $G - \{w_2, w_4, \dots\}$.
- (d) $\{w_2, w_4, \dots\}$ is a homogeneous set in $G - \{w_1, w_3, \dots\}$.

Since $\langle u, v \rangle$ is a u, v -weave, at least one u, v -weave with ≥ 2 vertices exists. Let $W = \langle w_1, \dots, w_n \rangle$ be a u, v -weave which maximizes $V(W)$. If $W = G$, then by Corollary 3.3 G is T_n or U_n , contradicting the assumptions of the theorem. So $|V(W)| < |V(G)|$.

Now, by Corollary 3.5, $W - \{w_i, w_{i+1}\}$ is isomorphic to $W - \{u, v\}$ for all $1 \leq i \leq n - 1$. In fact, because of the second sentence of Corollary 3.5 and (c) and (d) in the definition of a u, v -weave, we have that $G - \{w_i, w_{i+1}\}$ is isomorphic to $G - \{u, v\} = H$ for all $1 \leq i \leq n - 1$.

Let $H' = G - \{w_1, w_2\}$. Since H' is isomorphic to H , by our original assumption $H' + w_2$ must be decomposable. It follows that either $w_2 \in Z(H')$ or $w_2 \in V_x(H')$ for some $x \in V(H')$. Suppose $w_2 \in V_x(H')$ for some $x \in V(H')$. We thus have

- $x \Rightarrow \{w_3, w_5, \dots\} \setminus \{x\}$, and
- if $n \geq 4$, then $x \Rightarrow \{w_4, w_6, \dots\} \setminus \{x\}$ if W is FF or BF and $\{w_4, w_6, \dots\} \setminus \{x\} \Rightarrow x$ if W is FB or BB.

First suppose that $x \in V(G) \setminus V(W)$. Since $\{x, w_2\}$ is a homogeneous set of $H' + w_2 = G - w_1$ but it is not a homogeneous set of G , and $w_1 \rightarrow w_2$, we must have $x \rightarrow w_1$. Thus, $x \Rightarrow \{w_1, w_3, \dots\}$. Furthermore, by (d) in the definition of a u, v -weave, we have that $x \Rightarrow \{w_2, w_4, \dots\}$ if $x \Rightarrow \{w_4, w_6, \dots\}$ and $\{w_2, w_4, \dots\} \Rightarrow x$ if $\{w_4, w_6, \dots\} \Rightarrow x$. Finally, by Proposition 2.6, $\{x, w_2, w_4, \dots\}$ is a homogeneous set in $G - \{w_1, w_3, \dots\}$. Thus, $\langle x, w_1, \dots, w_n \rangle$ is a u, v -weave. This contradicts the maximality of W , so $x \notin V(G) \setminus V(W)$.

Now suppose $x \in V(W)$. We cannot have $x = w_i$ for any even $i \geq 4$ because $x \Rightarrow \{w_3, w_5, \dots\} \setminus \{x\}$ but $w_i \rightarrow w_3$ for all even $i \geq 4$. So $x = w_i$ for some odd $i \geq 3$. Now, since $w_2 \in V_{w_i}(H')$, for every $v \in V(G) \setminus V(W)$ we have $d_{vw_2} = d_{vw_i}$. Thus, since i is odd, from (c) and (d) in the definition of a u, v -weave we have that $V(W)$ is a homogeneous set of G . Since $2 \leq |V(W)| < |V(G)|$ as mentioned earlier, this contradicts the primeness of G .

Thus, we cannot have $w_2 \in V_x(H')$ for any $x \in V(H')$. So $w_2 \in Z(H')$. By symmetry, we also have that $w_{n-1} \in Z(H'')$, where $H'' = G - \{w_{n-1}, w_n\}$. If $n = 2$, then we have $w_1, w_2 \in Z(G - \{w_1, w_2\})$, and hence $G - \{w_1, w_2\}$ is a homogeneous set

of G , a contradiction since $|V(G)| \geq 5$. So assume $n \geq 3$. Then since $w_2 \rightarrow w_3$ and $w_2 \in Z(H')$, we have $w_2 \Rightarrow V(H')$. Similarly since $w_{n-2} \rightarrow w_{n-1}$ and $w_{n-1} \in Z(H'')$, we have $V(H'') \Rightarrow w_{n-1}$. In particular, we have $w_2 \Rightarrow V(G) \setminus V(W) \Rightarrow w_{n-1}$. If n is odd, then (c) and (d) from the definition of a u, v -weave imply that $\{w_2, w_4, \dots\} \Rightarrow V(G) \setminus V(W) \Rightarrow \{w_2, w_4, \dots\}$, a contradiction since $V(G) \setminus V(W)$ is nonempty. So n is even, and (c) and (d) imply that

$$\{w_2, w_4, \dots\} \Rightarrow V(G) \setminus V(W) \Rightarrow \{w_1, w_3, \dots\}.$$

Thus, $V(G) \setminus V(W)$ is a homogeneous set of G , and as we showed before it is nonempty. Hence, $|V(G) \setminus V(W)| = 1$. Let $\{v\} = V(G) \setminus V(W)$. Then $G = W + v$, and by Corollary 3.3, G must be T_{n+1} , U_{n+1} , or W_{n+1} . This is a contradiction, completing the proof of Claim 1. \square

Proof of Claim 2. We first prove the following general proposition.

Proposition 3.6. *Let G be a prime tournament and let $u \in V(G)$. Let H_1, H_2 be prime subtournaments of G such that $V(H_1) \cup V(H_2) = V(G) \setminus \{u\}$, $H_1 + u$ and $H_2 + u$ are decomposable, and the subtournament $H_{1,2}$ induced on $V(H_1) \cap V(H_2)$ is prime with $|V(H_{1,2})| \geq 3$. Then either $u \in V_x(H_1)$ for some $x \in V(H_1) \setminus V(H_2)$ or $u \in V_y(H_2)$ for some $y \in V(H_2) \setminus V(H_1)$.*

Proof. Since H_1 is prime and $H_1 + u$ is decomposable, we have either $u \in Z(H_1)$ or $u \in V_x(H_1)$ for some $x \in H_1$, and likewise for H_2 .

First, suppose $u \in Z(H_1)$ and $u \in Z(H_2)$. Then $V(H_1)$ is a homogeneous set of $H_1 + u$ and $V(H_2)$ is a homogeneous set of $H_2 + u$. Since $V(H_1) \cap V(H_2) \neq \emptyset$ by assumption, by Proposition 2.6 we have that $V(H_1) \cup V(H_2) = V(G) \setminus \{u\}$ is a homogeneous set of G . This contradicts the primeness of G , so we cannot have this case.

Next, suppose $u \in Z(H_1)$ and $u \in V_y(H_2)$ for some $y \in H_2$. If $y \in V(H_2) \setminus V(H_1)$ then we are done, so assume $y \in V(H_1)$, and hence $y \in V(H_{1,2})$. Now, $\{u, y\}$ is a homogeneous set of $H_2 + u$ and $V(H_1)$ is a homogeneous set of $H_1 + u$; restricting to the subtournament $H_{1,2} + u$, we have that $\{u, y\}$ and $V(H_{1,2})$ are homogeneous sets of $H_{1,2} + u$. Applying Proposition 2.5, we have that $V(H_{1,2}) \setminus \{y\}$ is a homogeneous set of $H_{1,2} + u$, and hence $V(H_{1,2}) \setminus \{y\}$ is a homogeneous set of $H_{1,2}$. Since $|V(H_{1,2})| \geq 3$, this contradicts the primeness of $H_{1,2}$.

Finally, suppose $u \in V_x(H_1)$ for some $x \in V(H_1)$ and $u \in V_y(H_2)$ for some $y \in V(H_2)$. If either $x \in V(H_1) \setminus V(H_2)$ or $y \in V(H_2) \setminus V(H_1)$ then we are done, so assume $x, y \in V(H_{1,2})$. If $x = y$, then $\{u, x\}$ is a homogeneous set of both $H_1 + u$ and $H_2 + u$, so by Proposition 2.2, $\{u, x\}$ is a homogeneous set of G , a contradiction. If $x \neq y$, then restricting to $H_{1,2} + u$ and applying Proposition 2.6, we have that $\{u, x, y\}$ is a homogeneous set of $H_{1,2} + u$, so $\{x, y\}$ is a homogeneous set of $H_{1,2}$. As before, this is a contradiction, which completes the proof. \square

We can now prove Claim 2. Assume the hypotheses of Claim 2. Let H be a subtournament of G which is isomorphic to T_n , U_n , or W_n . Let H' and H'' be prime

subtournaments of H with $n - 2$ and $n - 4$ vertices, respectively. In other words, if H is isomorphic to T_n , then H' is isomorphic to T_{n-2} and H'' is isomorphic to T_{n-4} ; likewise for U_n and W_n . We wish to prove there is a prime subtournament of G with $n - 1$ vertices that has a subtournament isomorphic to H' .

Suppose the contrary. Let $\{u\} = V(G) \setminus V(H)$. By Proposition 3.2, we can write H as $W + v$, where $W = \langle w_1, \dots, w_{n-1} \rangle$ is a weave and v is as in Proposition 3.2. For distinct integers $1 \leq i_1, \dots, i_r \leq n - 1$, let $H_{i_1, \dots, i_r} = H - \{w_{i_1}, \dots, w_{i_r}\}$. Note that $H_{i, i+1}$ is isomorphic to H' for all $1 \leq i \leq n - 2$. Hence, by assumption, $H_{i, i+1} + u$ must be decomposable for all $1 \leq i \leq n - 2$.

Now, for distinct integers $1 \leq i, j \leq n - 1$, we define $x_{i,j} \in V(H_{i,j}) \cup \{\infty\}$ as follows: If $u \in V_x(H_{i,j})$ for some $x \in V(H_{i,j})$, then let $x_{i,j} = x$; otherwise, let $x_{i,j} = \infty$. By Proposition 2.8(i), $x_{i,j}$ is well-defined. Now, suppose $1 \leq i, j \leq n - 2$ are integers such that $\{i, i + 1\} \cap \{j, j + 1\} = \emptyset$. Then $H_{i, i+1, j, j+1}$ is isomorphic to H'' , and is hence prime. Thus, applying Proposition 3.6 with $H_{i, i+1}$ as H_1 and $H_{j, j+1}$ as H_2 , we have that for all such i, j , either $x_{i, i+1} \in \{w_j, w_{j+1}\}$ or $x_{j, j+1} \in \{w_i, w_{i+1}\}$. We will denote this fact as (*).

Applying (*) with $i = 1$ and $j = n - 2$, we have that either $x_{1,2} \in \{w_{n-2}, w_{n-1}\}$ or $x_{n-2, n-1} \in \{w_1, w_2\}$. Without loss of generality, assume

$$x_{1,2} \in \{w_{n-2}, w_{n-1}\}.$$

Now, applying (*) with $i = 1$ and $j = 3$, we have that either $x_{1,2} \in \{w_3, w_4\}$ or $x_{3,4} \in \{w_1, w_2\}$. Since $n \geq 6$ and $x_{1,2} \in \{w_{n-2}, w_{n-1}\}$, we must have

$$x_{3,4} \in \{w_1, w_2\}.$$

Finally, applying (*) with $i = 3$ and $j = 5$, we have that either $x_{3,4} \in \{w_5, w_6\}$ or $x_{5,6} \in \{w_3, w_4\}$. Since we already have $x_{3,4} \in \{w_1, w_2\}$, we must have

$$x_{5,6} \in \{w_3, w_4\}.$$

Now, by the definition of $x_{i,j}$, we have that $\{u, x_{1,2}\}$, $\{u, x_{3,4}\}$, and $\{u, x_{5,6}\}$ are homogeneous sets of $H_{1,2}$, $H_{3,4}$, and $H_{5,6}$, respectively. Restricting to the subtournament $H_{1,2,3,4} + \{u, x_{1,2}, x_{3,4}\} = H_{1,2,3,4} + \{u, x_{3,4}\}$ (the equality holds because $x_{1,2} \in \{w_{n-2}, w_{n-1}\} \subseteq V(H_{1,2,3,4})$), we have that $\{u, x_{1,2}\}$ and $\{u, x_{3,4}\}$ are homogeneous sets of $H_{1,2,3,4} + \{u, x_{3,4}\}$, and hence by Proposition 2.6 and restriction, we have that $\{x_{1,2}, x_{3,4}\}$ is a homogeneous set of $H_{1,2,3,4} + x_{3,4}$. Similarly, $\{x_{3,4}, x_{5,6}\}$ is a homogeneous set of $H_{3,4,5,6} + x_{5,6}$.

Let $i_{1,2}, i_{3,4}, i_{5,6}$ be the integers such that $x_{1,2} = w_{i_{1,2}}$, $x_{3,4} = w_{i_{3,4}}$, and $x_{5,6} = w_{i_{5,6}}$. Then $\{w_{i_{1,2}}, w_{i_{3,4}}\}$ is a homogeneous set of $H_{1,2,3,4} + w_{i_{3,4}}$. In particular, we have $d_{vw_{i_{1,2}}} = d_{vw_{i_{3,4}}}$. Hence, by the definition of v , $i_{1,2}$ and $i_{3,4}$ have the same parity. Similarly, $i_{3,4}$ and $i_{5,6}$ have the same parity, so $i_{1,2}$, $i_{3,4}$, and $i_{5,6}$ all have the same parity. Suppose these numbers are even. Then $i_{1,2} = n - 1$ and $i_{3,4} = 2$. However, we have $w_2 \rightarrow w_{n-2}$ and $w_{n-2} \rightarrow w_{n-1}$ in $H_{1,2,3,4} + w_{i_{3,4}}$, which contradicts the fact that $\{w_{i_{1,2}}, w_{i_{3,4}}\} = \{w_2, w_{n-1}\}$ is a homogeneous set of $H_{1,2,3,4} + w_{i_{3,4}}$. Similarly, if

the numbers are odd, then $i_{3,4} = 1$ and $i_{5,6} = 3$, but $w_1 \rightarrow w_2$ and $w_2 \rightarrow w_3$ in $H_{3,4,5,6} + w_{i_{5,6}}$, contradicting the fact that $\{w_{i_{3,4}}, w_{i_{5,6}}\} = \{w_1, w_3\}$ is a homogeneous set of $H_{3,4,5,6} + w_{i_{5,6}}$. This completes the proof. \square

3.3 An application to D_4 -free tournaments

We conclude this section by using Theorem 3.1 to give a simple proof of a structural theorem. Let D_4 be the tournament on 4 vertices consisting of a cyclic triangle C and a vertex v with $v \Rightarrow C$. Let D_4^* be the tournament formed from D_4 by reversing all of its edges. A well-known theorem classifies all tournaments that exclude both D_4 and D_4^* .

Theorem 3.7 (Gnanvo and Ille [5], Lopez and Rauzy [7]). *A prime tournament G with $|V(G)| \geq 5$ does not have a subtournament isomorphic to D_4 or D_4^* if and only if G is T_n for some odd $n \geq 5$.*

In fact, this theorem still holds true if we replace excluding both D_4 and D_4^* with excluding only D_4 . We give a short proof of this result without relying on Theorem 3.7 itself.

Theorem 3.8. *A prime tournament G with $|V(G)| \geq 3$ does not have a subtournament isomorphic to D_4 if and only if G is T_n for some odd $n \geq 3$.*

Proof. To prove one direction, note that if G is T_n for some odd $n \geq 3$, then the outneighborhood of every vertex $v \in V(G)$ forms a transitive tournament. Since D_4 consists of a vertex whose outneighborhood is a cyclic triangle, G does not have a subtournament isomorphic to D_4 .

We now prove the other direction. Suppose G is a prime tournament with $|V(G)| \geq 3$ that does not have a subtournament isomorphic to D_4 . If $|V(G)| < 5$, then G is T_3 and we are done. So assume $|V(G)| \geq 5$. G cannot be U_n or W_n for any n because these tournaments have subtournaments isomorphic to D_4 , and if G is T_n for some n then we are done. So assume G is not T_n , U_n , or W_n for any n .

By Corollary 2.10, G has a prime subtournament H_5 with 5 vertices, and since U_n and W_n each have D_4 subtournaments, H_5 must be T_5 . If $G = H_5$, we are done; otherwise, by Theorem 2.9, G has a prime subtournament H with 6 vertices that has a subtournament isomorphic to T_5 . Let H' be a subtournament of H isomorphic to T_5 , and let $\{u\} = V(H) \setminus V(H')$. Let the vertices of H' be v_1, \dots, v_5 as in Definition 2.7.

Since H is prime, we have $u \notin Z(H')$. Hence, there must be some $1 \leq i \leq 5$ such that $u \rightarrow v_i$ and $v_{i+1} \rightarrow u$ (where the indices are taken modulo 5). Without loss of generality, assume $u \rightarrow v_2$ and $v_3 \rightarrow u$. Then $\{u, v_2, v_3\}$ forms a cyclic triangle. Since $v_1 \Rightarrow \{v_2, v_3\}$, we must have $u \rightarrow v_1$ (or else $\{v_1, u, v_2, v_3\}$ would form a D_4). Now, $\{v_1, v_2, v_4\}$ forms a cyclic triangle, and $u \Rightarrow \{v_1, v_2\}$, so we must have $v_4 \rightarrow u$. Thus, $\{v_3, v_4\} \Rightarrow u \Rightarrow \{v_1, v_2\}$. But then $\{u, v_5\}$ is a homogeneous set of H , which contradicts the primeness of H . This completes the proof. \square

Corollary 3.9. *A tournament G does not have a subtournament isomorphic to D_4 if and only if it can be written as $T_n(I^1, \dots, I^n)$ or $I_2(T_n(I^1, \dots, I^n), I)$, where $n \geq 1$ is odd, I_2 has the standard ordering of vertices, and I^1, \dots, I^n, I are transitive tournaments.*

Proof. If G can be written in one of the forms mentioned, then the outneighborhood of every vertex $v \in V(G)$ forms a transitive tournament, and hence G does not have a subtournament isomorphic to D_4 . This proves one direction.

Now, suppose G does not have a subtournament isomorphic to D_4 . If G has only one vertex, then we can write G as $T_1(T_1)$, and we are done. So assume $|V(G)| \geq 2$. Write G as $G'(H_1, \dots, H_n)$, where G' is a prime tournament with $|V(G')| \geq 2$. First, suppose that $|V(G')| \geq 3$. Then by Theorem 3.7, G' is T_n for some odd n . Now, if any of H_1, \dots, H_n , say H_i , contains a cyclic triangle, then this triangle forms a D_4 with any vertex in H_{i-1} (where the index is taken modulo n). So H_i is transitive for all i , and thus G is of the first form in the theorem, as desired.

Now suppose that $|V(G')| = 2$, and hence $G' = P_2$. Then G is of the form $I_2(H_1, H_2)$. If H_2 contains a cyclic triangle, then this triangle forms a D_4 with any vertex in H_1 ; thus, H_2 is transitive. Now, write G as $P_2(H_1, I_k)$ in such a way that k is maximal. If $|V(H_1)| = 1$ then $H_1 = I_1$ and we are done. Otherwise, write H_1 as $H'_1(K_1, \dots, K_m)$, where H'_1 is prime and $|V(H'_1)| \geq 2$. If $|V(H'_1)| \geq 3$, then since H_1 is D_4 -free, we have as before that H'_1 is T_m for odd m and each K_i is transitive. Thus G is of the second form in the theorem, as desired. Otherwise, H'_1 is P_2 , and we have as before that H_1 is $P_2(K_1, I_j)$ for some j . But then G can be written as $P_2(K_1, I_{j+k})$, contradicting the maximality of k . This completes the proof. \square

4 Cyclic triangles in prime tournaments

4.1 Triangle-connectivity

Let G be a tournament. Say that two cyclic triangles C, C' in G are *adjacent* if they share exactly two vertices. We say that two cyclic triangles C, C' are *triangle-connected* to each other if there is a sequence of cyclic triangles C_1, \dots, C_n , $n \geq 1$, such that $C_1 = C$, $C_n = C'$, and C_i is adjacent to C_{i+1} for all $1 \leq i \leq n-1$. We can ask the following question: when are all the cyclic triangles of a tournament triangle-connected to each other?

Call a tournament *triangle-connected* if it is strongly connected and any two cyclic triangles in the tournament are triangle-connected. In general, nontrivial homogeneous sets that contain cyclic triangles can stop a tournament from being triangle-connected; to see this, let G be a strongly connected tournament and suppose X is a nontrivial homogeneous set of G . Then a cyclic triangle all of whose vertices are in X cannot be triangle-connected to a cyclic triangle which has a vertex not in X , because if there were indeed a sequence of adjacent cyclic triangles connecting two such triangles, there must be some triangle in the sequence with two vertices $\{v_1, v_2\}$

in X and one vertex v_3 not in X , a contradiction since X is a homogeneous set so either $v_3 \Rightarrow \{v_1, v_2\}$ or $\{v_1, v_2\} \Rightarrow v_3$.

One could then ask whether a tournament that is strongly connected and which has no nontrivial homogeneous sets that contain cyclic triangles is triangle-connected. The answer is positive, and follows from the next theorem.

Theorem 4.1. *If G is a strongly connected prime tournament, then it is triangle-connected.*

Corollary 4.2. *A strongly connected tournament is triangle-connected if and only if it has no nontrivial homogeneous sets that contain cyclic triangles.*

To see that the corollary follows from the theorem, let G be a strongly connected tournament. First, suppose that $X \subseteq V(G)$ is a nontrivial homogeneous set of G which contains a cyclic triangle C . Since X is nontrivial, let $v \in V(G) \setminus X$. Since G is strongly connected, v is a vertex of some cyclic triangle C' in G . Then as noted before, C and C' are not triangle-connected to each other, so G is not triangle-connected. This proves one direction of the corollary.

For the other direction, suppose G has no nontrivial homogeneous sets that contain cyclic triangles. If G has one vertex then it is trivially triangle-connected, so assume $|V(G)| \geq 2$. Write G as $G'(H_1, \dots, H_n)$ where G' is prime (and strongly connected, since G is). Since each $V(H_i)$ is a homogeneous set of G , none of the H_i contain a cyclic triangle. Hence, every cyclic triangle of G has vertices $\{v_i, v_j, v_k\}$ for some distinct i, j, k such that $v_i \in V(H_i)$, $v_j \in V(H_j)$, and $v_k \in V(H_k)$. Call a cyclic triangle a $\{i, j, k\}$ -triangle if it has vertices in $V(H_i)$, $V(H_j)$, and $V(H_k)$. It is easy to see that for each set of distinct i, j, k , all $\{i, j, k\}$ -triangles are triangle-connected to each other. Also, since G' is prime and strongly connected, by Theorem 4.1, every $\{i, j, k\}$ -triangle is connected to a $\{i', j', k'\}$ -triangle for every $\{i', j', k'\}$ for which such a triangle exists. Thus, every cyclic triangle of G is triangle-connected to each other, completing the proof of the corollary.

4.2 Proof of Theorem 4.1

To prove the theorem, we use Theorem 3.1 (or Theorem 2.11) and induction. Because Theorem 3.1 does not hold for T_n , U_n , or W_n , we will treat these cases separately.

Proof for T_n , U_n , and W_n . Let G be T_n , U_n , or W_n for some odd $n \geq 1$. We prove that G is triangle-connected. The case $n = 1$ is trivial, so assume $n \geq 3$. As in the proof of Theorem 3.1, write G as $W + v$, where $W = \langle w_1, \dots, w_{n-1} \rangle$ is a weave and v is as in Proposition 3.2.

It suffices to prove that every cyclic triangle in G is triangle-connected to the cyclic triangle with vertices $\{v, w_1, w_{n-1}\}$. Given distinct vertices $x, y, z \in V(G)$, let $C(x, y, z)$ be the triangle with vertices $\{x, y, z\}$ (the triangle may or may not be cyclic). Every cyclic triangle in G is of one of the following forms:

- (a) $C(v, w_i, w_j)$, where $i < j$ and i is odd and j is even. (All such triangles are cyclic.)
- (b) $C(w_i, w_j, w_k)$ where $i < j < k$ and j has opposite parity from i and k . (Such a triangle might not be cyclic depending on the type of weave.)

Let C be a cyclic triangle in G . If $C = C(v, w_i, w_j)$ as in (a), we have that C is adjacent to or the same as the cyclic triangle $C(v, w_1, w_j)$, and this triangle is adjacent to or the same as $C(v, w_1, w_{n-1})$. Thus, C is triangle-connected to $C(v, w_1, w_{n-1})$, as desired.

Now suppose $C = C(w_i, w_j, w_k)$ as in (b). Then one of the triangles $C(v, w_i, w_j)$, $C(v, w_j, w_k)$ is of type (a), and hence this triangle is cyclic and, as above, it is triangle-connected to $C(v, w_1, w_{n-1})$. Since C is adjacent to this cyclic triangle, C is also triangle-connected to $C(v, w_1, w_{n-1})$, as desired. This completes the proof for T_n , U_n , and W_n . □

Now, let G be strongly connected and prime. We will prove that G is triangle-connected by induction on $|V(G)|$. All base cases $|V(G)| \leq 5$ were covered in the proof for T_n , U_n , and W_n . Suppose $|V(G)| \geq 6$, and that the theorem holds for tournaments with $|V(G)| - 1$ vertices. If G is T_n , U_n , or W_n for any n , then from the above argument we are done. Assume G is not one of these tournaments. By Theorem 3.1 (or Theorem 2.11), there is a prime subtournament H of G with $|V(G)| - 1$ vertices. Let $\{u\} = V(G) \setminus V(H)$. Since H is triangle-connected by the inductive hypothesis, to show that G is triangle-connected it suffices to show that every cyclic triangle of G with u as a vertex is triangle-connected to a cyclic triangle in H .

Suppose the contrary. Let \mathcal{C} denote the set of cyclic triangles of G that have u as a vertex. Let $C = C(u, v_A, v_B)$ be a cyclic triangle in \mathcal{C} that is not triangle-connected to a cyclic triangle in H , where $u \rightarrow v_A \rightarrow v_B \rightarrow u$. We will prove the following two claims:

Claim 1. C is triangle-connected to every cyclic triangle in \mathcal{C} .

Claim 2. There is a cyclic triangle in \mathcal{C} which is triangle-connected to a cyclic triangle in H .

These two claims clearly contradict the definition of C , which will complete the proof.

Proof of Claim 1. This proof is due to P. Seymour. Let $A = A_G(u)$ and $B = B_G(u)$, so $v_A \in A$, $v_B \in B$, and $A \cup B = V(H)$. We associate each cyclic triangle $C(u, x_A, x_B)$ in \mathcal{C} , where $x_A \in A$ and $x_B \in B$, with the directed edge $x_A x_B$; this gives a one-to-one correspondence between the triangles in \mathcal{C} and the edges from A to B . Let \hat{H} be the undirected graph with vertices $V(H)$ and an undirected edge xy for each directed edge xy in H with $x \in A$ and $y \in B$. Then two triangles in \mathcal{C} are triangle-connected if and only if their associated edges in \hat{H} are in the same connected component of \hat{H} .

Let \hat{H}' be the connected component of \hat{H} containing $v_A v_B$, and let $A' = A \cap V(\hat{H}')$ and $B' = B \cap V(\hat{H}')$. Suppose there is a vertex $v \in A$ which is not in A' . We claim that $\{u\} \cup A' \cup B' \Rightarrow v$. Because v is not adjacent in \hat{H} to any vertex in B' , we must have $B' \Rightarrow v$ in the tournament H . Moreover, if there is a vertex $x_{A'} \in A'$ such that $v \rightarrow x_{A'}$, then $x_{A'} \rightarrow x_{B'}$ for some $x_{B'} \in B'$, and $C(v, x_{A'}, x_{B'})$ is a cyclic triangle in H . C is triangle-connected to this triangle, which contradicts the fact that C is not triangle-connected to a triangle in H . Thus, we must have $A' \Rightarrow v$. Finally, $u \rightarrow v$ by the definition of A , so altogether we have $\{u\} \cup A' \cup B' \Rightarrow v$, as claimed. Similarly, if $v \in B$ and $v \notin B'$, we have $v \Rightarrow \{u\} \cup A' \cup B'$. It follows that $\{u\} \cup A' \cup B'$ is a homogeneous set of H . Since this set contains $\{u, v_A, v_B\}$ and G is prime, we must have $\{u\} \cup A' \cup B' = V(G)$, and hence the connected component \hat{H}' is all of \hat{H} . Thus, C is triangle-connected to every triangle in \mathcal{C} , as desired. \square

Proof of Claim 2. Suppose that no cyclic triangle in \mathcal{C} is triangle-connected to a cyclic triangle in H . Since every four-vertex tournament is decomposable, for every cyclic triangle $C(x, y, z)$ in H , we have that either $u \in Z(C(x, y, z))$ or $u \in V_v(C(x, y, z))$ for some $v \in \{x, y, z\}$. If the latter case, say $u \in V_x(C(x, y, z))$, then $C(u, y, z)$ is a cyclic triangle, and it is adjacent to $C(x, y, z)$, contradicting our original assumption. Thus, $u \in Z(C')$ for every cyclic triangle C' in H , and hence for every such cyclic triangle either $u \Rightarrow C'$ or $C' \Rightarrow u$. However, by the inductive hypothesis, any two cyclic triangles in H can be connected to each other by a sequence of adjacent cyclic triangles, so in fact we have that either $u \Rightarrow C'$ for every cyclic triangle C' in H or $C' \Rightarrow u$ for every cyclic triangle C' in H . Since H is strongly connected, every vertex of H belongs to a cyclic triangle of H . Thus, $V(H)$ is a homogeneous set of G , contradicting the fact that G is prime. This completes the proof. \square

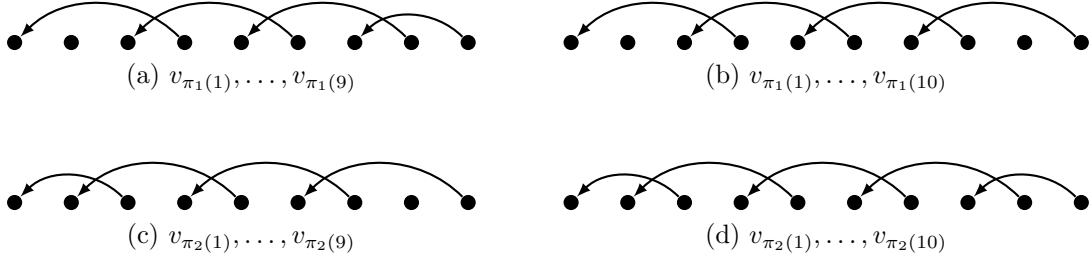
5 Matching tournaments

5.1 Matching orderings

We now focus on a somewhat different topic. Given a tournament G and an ordering v_1, \dots, v_n of its vertices, an edge $v_j v_i$ with $j > i$ is called a *backedge*. A *matching ordering* of a tournament G is an ordering of its vertices such that every vertex is the head or tail of at most one backedge. A tournament with at least one matching ordering is called a *matching tournament*. In Subsections 5.1 through 5.4, we will deal with the question of how many matching orderings a matching tournament can have. Subsection 5.5 will deal with minimal non-matching tournaments.

In general, a matching tournament can have many matching orderings. For example, I_n has at least $2^{n/2}$ matching orderings: Let v_1, \dots, v_n be the standard ordering of $V(I_n)$. Let S_n denote the symmetric group on $\{1, 2, \dots, n\}$, and let τ_i denote the transposition $(i \ i+1)$. Then $v_{\pi(1)}, \dots, v_{\pi(n)}$ is a matching ordering of I for all permutations $\pi \in S_n$ of the form $\pi = \tau_1^{e_1} \tau_3^{e_3} \tau_5^{e_5} \cdots \tau_{2\lfloor n/2 \rfloor - 1}^{e_{2\lfloor n/2 \rfloor - 1}}$ and $\pi = \tau_2^{e_2} \tau_4^{e_4} \tau_6^{e_6} \cdots \tau_{2\lfloor n/2 \rfloor}^{e_{2\lfloor n/2 \rfloor}}$, where $e_i = 0$ or 1 for all i .

An example of an infinite family of prime tournaments with more than one matching ordering is as follows: For $n \geq 1$, let P_n be the tournament with vertices v_1, \dots, v_n such that $v_i \rightarrow v_j$ if $j - i \geq 2$, and $v_{i+1} \rightarrow v_i$ for all $1 \leq i \leq n - 1$. For all $n \neq 4$, P_n is prime. (Note that P_2 is I_2 , P_3 is the cyclic triangle, and P_5 is W_5 .) Let $\pi_1 = \tau_1 \tau_3 \dots \tau_{2\lfloor n/2 \rfloor - 1}$ and $\pi_2 = \tau_2 \tau_4 \dots \tau_{2\lfloor n/2 \rfloor - 2}$, where $\tau_i = (i \ i + 1)$ as before. Then $v_{\pi_1(1)}, \dots, v_{\pi_1(n)}$ and $v_{\pi_2(1)}, \dots, v_{\pi_2(n)}$ are both matching orderings of P_n , and these orderings are distinct if $n > 1$. Below we draw these two matching orderings for both odd and even examples of n ; only backedges are shown.



Despite these examples, having more than one matching ordering is in fact a very strict condition. One reason for this is a simple consideration of vertex degrees, which gives the following fact.

Proposition 5.1. *Let v_1, \dots, v_n be a matching ordering of G . If v is a vertex of G with indegree b , then v is either v_b , v_{b+1} , or v_{b+2} . Moreover, $v = v_b$ if and only if it is the head of a backedge, $v = v_{b+2}$ if and only if it is the tail of a backedge, and $v = v_{b+1}$ if and only if it is not an end of a backedge.*

Proof. Since $v_i \rightarrow v_j$ for each $i < j$ except for backedges $v_j v_i$, and every vertex is the end of at most one backedge, we have $|B_G(v_i)| = i$ if v_i is the head of a backedge, $|B_G(v_i)| = i - 2$ if v_i is the tail of a backedge, and $|B_G(v_i)| = i - 1$ if v_i is not an end of a backedge. \square

We will show that, in a sense, P_n is the only “fundamental” example of a tournament with more than one matching ordering. More specifically, we will show that the only reason a tournament might have more than one matching ordering is that it has a homogeneous set on which the induced subtournament is P_n for $n > 1$.

5.2 Statement and proof of theorem

Before stating the exact theorem, we introduce several definitions and facts. Let $X = \{b, b + 1, \dots, a\}$ be a nonempty set of consecutive integers. Define σ_X to be the permutation on X as follows.

- If $|X|$ is odd, $\sigma_X = (b \ b + 2 \ b + 4 \ \dots \ a - 2 \ a \ a - 1 \ a - 3 \ \dots \ b + 1)$.
- If $|X|$ is even, $\sigma_X = (b \ b + 2 \ b + 4 \ \dots \ a - 1 \ a \ a - 2 \ a - 4 \ \dots \ b + 1)$.

(If $|X| = 1$, σ_X is the identity, and if $|X| = 2$, σ_X is a transposition.) Defining $\pi_1 = \tau_1\tau_3 \dots \tau_{2\lfloor n/2 \rfloor - 1}$ and $\pi_2 = \tau_2\tau_4 \dots \tau_{2\lfloor n/2 \rfloor - 2}$ as in the previous subsection, we also have that $\sigma_{\{1, \dots, n\}} = \pi_1\pi_2$.

If X is as above and $|X| = 4$, define $\tau_X = (b \ b+2)(b+1 \ b+3)$. We have $\tau_{\{1,2,3,4\}} = \pi_2\pi_1\pi_2$, where we use $n = 4$ in the definition of π_1 and π_2 .

Finally, suppose $\pi \in S_n$. Given a transitive tournament G with the standard ordering v_1, \dots, v_n of $V(G)$, let πG denote the ordering $v_{\pi(1)}, \dots, v_{\pi(n)}$ of $V(G)$. Similarly, given a tournament G isomorphic to P_n , if $n \neq 3$ then there is a unique ordering v_1, \dots, v_n of its vertices which satisfies the definition of P_n given previously. Let πG denote the ordering $v_{\pi(1)}, \dots, v_{\pi(n)}$ of $V(G)$. If G is P_3 , then there is no unique ordering satisfying the definition, so we will say that an ordering is πG if it is $v_{\pi(1)}, v_{\pi(2)}, v_{\pi(3)}$ for *some* ordering v_1, v_2, v_3 of $V(G)$ that satisfies the definition of P_n . If $|V(G)| = 2$, then G is isomorphic to both I_2 and P_2 but the two above definitions give different orderings for πG , so in this case we let πG denote any ordering of $V(G)$.

Note that $|\sigma_X(x) - x| \leq 2$ and $|\tau_X(x) - x| \leq 2$ for all $x \in X$. Also, as we noted in the previous subsection, if G is transitive or isomorphic to P_n , then $\pi_1 G$ and $\pi_2 G$ are matching orderings of G . If G is P_4 , we can check that $\pi_2\pi_1 G$ is also a matching ordering of G . The next two propositions can be thought of as converses to these facts.

Proposition 5.2. *Let σ be a permutation of $\{1, 2, \dots, n\}$ which can be written as a cycle $(x_1 \ x_2 \ \dots \ x_k)$, $k \geq 1$. If $|\sigma(x) - x| \leq 2$ for all $x \in \{1, \dots, n\}$, then either*

- $X = \{x_1, \dots, x_k\}$ is a set of consecutive integers and $\sigma = \sigma_X$ or σ_X^{-1} , or
- $k = 2$ and $|x_1 - x_2| = 2$.

Proposition 5.3. *Let G be a tournament with matching ordering v_1, \dots, v_n , and let $X = \{1, \dots, n\}$. Suppose $\sigma \in S_n$ such that $v_{\sigma(1)}, \dots, v_{\sigma(n)}$ is a matching ordering of G . Then the following hold.*

- If $\sigma = \sigma_X$, then G is either transitive or isomorphic to P_n , and the ordering v_1, \dots, v_n is $\pi_2 G$.
- If $\sigma = \sigma_X^{-1}$, then G is either transitive or isomorphic to P_n , and the ordering v_1, \dots, v_n is $\pi_1 G$.
- If $n = 4$ and $\sigma = \tau_X$, then G is isomorphic to P_4 , and the ordering v_1, \dots, v_n is either $\pi_2 G$ or $\pi_2\pi_1 G$.

For the sake of pacing, we defer the proofs of these propositions until after the next subsection.

We now state and prove the main theorem.

Theorem 5.4. *Let G be a tournament and v_1, \dots, v_n be a matching ordering of G . Suppose there is a permutation $\pi \in S_n$ such that $v_{\pi(1)}, \dots, v_{\pi(n)}$ is also a matching ordering of G . Then $\{1, \dots, n\}$ can be partitioned into sets X_1, \dots, X_r , where each X_i is a set of consecutive integers, such that*

- $\pi = \sigma_1 \sigma_2 \dots \sigma_r$ where for each $1 \leq i \leq r$, σ_i is either σ_{X_i} , $\sigma_{X_i}^{-1}$, or τ_{X_i} .
- For each $1 \leq i \leq r$, the set $V_i = \{v_x : x \in X_i\}$ is a homogeneous set of G .
- For each $1 \leq i \leq r$, the subtournament H_i induced on V_i is either transitive or isomorphic to $P_{|V_i|}$. Furthermore, the ordering induced on V_i by the ordering v_1, \dots, v_n is given by σ_i as in Proposition 5.3.

Proof. Suppose vertex $v_{\pi(i)}$ has indegree b . By Proposition 5.1, for any matching ordering u_1, \dots, u_n of G , we have that $v_{\pi(i)}$ appears in the b -th, $(b+1)$ -th, or $(b+2)$ -th position. In particular, since $v_{\pi(i)}$ appears in the $\pi(i)$ -th position of v_1, \dots, v_n and in the i -th position of $v_{\pi(1)}, \dots, v_{\pi(n)}$, we have $\pi(i), i \in \{b, b+1, b+2\}$. It follows that $|\pi(i) - i| \leq 2$ for all $1 \leq i \leq n$.

Now, we can write $\pi = \sigma'_1 \dots \sigma'_{r'}$ where $\sigma'_1, \dots, \sigma'_{r'}$ are disjoint cycles whose orbits form a partition of $\{1, \dots, n\}$. (Some of the σ'_i may have orbits of size one.) Let X'_i be the orbit of σ'_i . Now, for each $1 \leq i \leq r'$, we have $|\sigma'_i(x) - x| \leq 2$ for all $1 \leq x \leq n$. Hence, by Proposition 5.2, for each $1 \leq i \leq r'$ either X'_i is a set of consecutive integers and $\sigma'_i = \sigma_{X'_i}^{\pm 1}$, or $X'_i = \{x, x+2\}$ for some x .

Suppose $X'_i = \{x, x+2\}$ for some i . The integer $x+1$ must belong to X'_j for some $j \neq i$. Either X'_j is a set of consecutive integers, or $X'_j = \{x-1, x+1\}$ or $\{x+1, x+3\}$. Suppose the former case; then since $x, x+2 \in X'_i$, we must have $X'_j = \{x+1\}$. Thus, $\pi(x+1) = x+1$. Now, consider the subtournament H induced on $\{v_x, v_{x+1}, v_{x+2}\}$. Using the orderings of $V(H)$ induced by the matching orderings v_1, \dots, v_n and $v_{\pi(1)}, \dots, v_{\pi(n)}$, we have that v_x, v_{x+1}, v_{x+2} and v_{x+2}, v_x, v_{x+1} are both matching orderings of H . However, there are only two three-vertex tournaments, and simple inspection shows that neither has two matching orderings satisfying this. This is a contradiction, so X'_j is not a set of consecutive integers.

Hence, $X'_j = \{x-1, x+1\}$ or $\{x+1, x+3\}$. Either way, $X'_i \cup X'_j$ is a set of four consecutive integers, and $\sigma'_i \sigma'_j = \tau_{X'_i \cup X'_j}$. We can pair up all the X'_i of the form $\{x, x+2\}$ in this manner. Replacing each such pair with their union and including all the X'_i that were originally sets of consecutive integers, we have in the end a partition $\{X_1, \dots, X_r\}$ of $\{1, \dots, n\}$ where each X_i is a set of consecutive integers, and $\pi = \sigma_1 \dots \sigma_r$ where σ_i is either $\sigma_{X_i}^{\pm 1}$ or τ_{X_i} , as desired.

Next, we wish to show that each $V_i = \{v_x : x \in X_i\}$ is a homogeneous set in G . If $|V_i| = 1$ then we are trivially done. So assume $|V_i| \geq 2$. Let $X_i = \{b, b+1, \dots, a\}$. We will show that if $b' < b$ then $v_{b'} \Rightarrow V_i$, and if $a' > a$ then $V_i \Rightarrow v_{a'}$, which will prove the claim. Let $b' < b$, and suppose there is some $x \in X_i$ such that $v_x \rightarrow v_{b'}$. Then v_x is the tail of the backedge $v_x v_{b'}$ in the ordering v_1, \dots, v_n . Now, since b' and x are in different cycle orbits of π , and these orbits are sets of consecutive integers, $v_x v_{b'}$ is also a backedge of $v_{\pi(1)}, \dots, v_{\pi(n)}$. Thus, v_x is also the tail of the backedge $v_x v_{b'}$ in $v_{\pi(1)}, \dots, v_{\pi(n)}$. By the second sentence of Proposition 5.1, v_x must therefore be in the same position in $v_{\pi(1)}, \dots, v_{\pi(n)}$ as it is in v_1, \dots, v_n . However, this is a contradiction, because $x \in X_i$ and σ_i has no fixed points in X_i for $|X_i| \geq 2$. We therefore have $v_{b'} \Rightarrow V_i$. The proof that $V_i \Rightarrow v_{a'}$ for $a' > a$ is analagous, so we have the desired claim.

Finally, to prove the last point, let $X_i = \{b, b+1, \dots, a\}$. The orderings induced on V_i by v_1, \dots, v_n and $v_{\pi(1)}, \dots, v_{\pi(n)}$ are v_b, v_{b+1}, \dots, v_a and $v_{\sigma_i(b)}, v_{\sigma_i(b+1)}, \dots, v_{\sigma_i(a)}$, respectively. These are matching orderings of H_i , so by Proposition 5.3, H_i is either transitive or isomorphic to $P_{|V_i|}$. Furthermore, the ordering v_b, v_{b+1}, \dots, v_a is given by σ_i as in Proposition 5.3, as desired. \square

5.3 Corollaries to Theorem 5.4

We state a few notable corollaries to the previous theorem.

Corollary 5.5. *Every prime tournament which is not P_n for any n has at most one matching ordering.*

Proof. Let G be a prime tournament which is not P_n for any n , and suppose v_1, \dots, v_n is a matching ordering of G . Let $\pi \in S_n$ such that $v_{\pi(1)}, \dots, v_{\pi(n)}$ is a matching ordering of G . Write $\pi = \sigma_1 \sigma_2 \dots \sigma_r$ as in Theorem 5.4. If $1 < |X_i| < |V(G)|$ for any i , then $V_i = \{v_x : x \in X_i\}$ is a nontrivial homogeneous set of G , a contradiction. So we must either have $|X_i| = 1$ for all i , or $r = 1$ and $X_1 = \{1, \dots, n\}$. If the former case, then π is the identity, so $v_{\pi(1)}, \dots, v_{\pi(n)}$ and v_1, \dots, v_n are the same ordering. If the latter case, then $V_1 = V(G)$ and so by the third point of Theorem 5.4, G is either transitive or P_n , contradicting our assumptions on G . Thus, G has at most one matching ordering. \square

Corollary 5.6. *P_n has exactly one matching ordering for $n = 1$, exactly three matching orderings for $n = 3, 4$, and exactly two matching orderings for all other n .*

Proof. We can check the cases $n = 1, 2, 3, 4$ by hand. Assume $n \geq 5$; thus, P_n is prime. Let v_1, \dots, v_n be a matching ordering of P_n , and suppose $\pi \in S_n$ such that $v_{\pi(1)}, \dots, v_{\pi(n)}$ is a matching ordering of P_n . By the same argument as in the previous proof, either $\pi = 1$ or $\pi = \sigma_X^e$, where $X = \{1, \dots, n\}$ and $e = \pm 1$. (We do not have $\pi = \tau_X$ because $|X| \geq 5$.) If $\pi = \sigma_X^e$, then there is only one possible value of e , because by Proposition 5.3, e is determined by whether v_1, \dots, v_n is $\pi_1 P_n$ or $\pi_2 P_n$. Thus, either $\pi = 1$ or $\pi = \sigma_X^e$ where there is only one possible value of e . It follows that P_n has at most two matching orderings. Since $\pi_1 P_n$ and $\pi_2 P_n$ are distinct matching orderings of P_n , P_n has exactly two matching orderings, as desired. (Note: Although P_3 is prime, the above argument does not work for P_3 because the way we have defined πP_3 , v_1, v_2, v_3 can be both $\pi_1 P_3$ and $\pi_2 P_3$, and hence we cannot determine e from Proposition 5.3.) \square

Corollary 5.7. *I_n has exactly F_n matching orderings, where $\{F_i\}_{i \geq 0}$ are the Fibonacci numbers defined by $F_0 = F_1 = 1$ and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$.*

Proof. Let v_1, \dots, v_n be the standard ordering of I_n . Suppose $\pi \in S_n$ such that $v_{\pi(1)}, \dots, v_{\pi(n)}$ is a matching ordering of I_n . For this π , let X_1, \dots, X_r be as in Theorem 5.4. We claim that $|X_i| \leq 2$ for all i . By Proposition 5.3, for each i the ordering induced on $V_i = \{v_x : x \in X_i\}$ by v_1, \dots, v_n is of the form $\pi_s I_{|V_i|}$, $\pi_s P_{|V_i|}$, or

$\pi_2\pi_1P_4$, where $s = 0$ or 1 . However, v_1, \dots, v_n has no backedges, and by inspection the only orderings that are of one of these forms and have no backedges are $\pi_s I_1$ (or $\pi_s P_1$) and $\pi_2 I_2$ (or $\pi_1 P_2$). Thus, $|X_i| \leq 2$ for all i , as claimed.

Conversely, for each partition $\{X_1, \dots, X_r\}$ of $\{1, \dots, n\}$ where each X_i is a set of consecutive integers of size 1 or 2, there is exactly one $\pi = \sigma_1 \cdots \sigma_r$ satisfying Theorem 5.4 which has $\{X_1, \dots, X_r\}$ as its associated partition, and it is not hard to see that $v_{\pi(1)}, \dots, v_{\pi(n)}$ is a matching ordering of I for this π . Hence, the number of matching orderings of I is equal to the number of ways to partition $\{1, \dots, n\}$ into sets of consecutive integers of size 1 or 2. We can prove by induction that this number is F_n . \square

5.4 Proofs of Propositions 5.2 and 5.3

We now give the proofs of Propositions 5.2 and 5.3.

Proof of Proposition 5.2. It is easy to see the proposition holds for $k = 1, 2$. Assume $k \geq 3$. Let $x = \min(x_1, x_2, \dots, x_k)$. We have both $|\sigma(x_i) - x_i| \leq 2$ and $|\sigma^{-1}(x_i) - x_i| \leq 2$ for all $1 \leq i \leq k$. In particular, since $x = \min(x_1, \dots, x_k)$, we have $\sigma(x), \sigma^{-1}(x) \in \{x+1, x+2\}$. Since the order of σ is $k \geq 3$, we have $\sigma(x) \neq \sigma^{-1}(x)$. Hence, $\{\sigma(x), \sigma^{-1}(x)\} = \{x+1, x+2\}$.

Suppose $\sigma^{-1}(x) = x+1$, so $\sigma(x) = x+2$. We will prove by induction that for all $0 \leq j \leq k-1$,

$$\sigma^{(-1)^j \lceil j/2 \rceil}(x) = x+j.$$

This gives the value of $\sigma^i(x)$ for k consecutive values of i , and these values coincide with $\sigma_X^i(x)$, where $X = \{x, x+1, \dots, x+k-1\}$. Since σ and σ_X are both cycles of order k , this will imply $\sigma = \sigma_X$, as desired.

The base case $j = 0$ is trivial, and the base cases $j = 1, 2$ are given by assumption. Let $3 \leq j \leq k-1$, and suppose that $\sigma^{(-1)^{j_0} \lceil j_0/2 \rceil}(x) = x+j_0$ for all $0 \leq j_0 \leq j-1$. In particular, for $j_0 = j-2$ we have

$$\begin{aligned} \sigma^{(-1)^{j-2} \lceil (j-2)/2 \rceil}(x) &= x+j-2 \\ \sigma^{(-1)^j (\lceil j/2 \rceil - 1)}(x) &= x+j-2 \\ \Rightarrow \sigma^{(-1)^j \lceil j/2 \rceil}(x) &= \sigma^{(-1)^j}(x+j-2). \end{aligned}$$

Now, $|\sigma^{(-1)^j}(x+j-2) - (x+j-2)| \leq 2$, so in particular, $\sigma^{(-1)^j}(x+j-2) \leq x+j$. Combined with the previous equality, we have

$$\sigma^{(-1)^j \lceil j/2 \rceil}(x) \leq x+j.$$

If $\sigma^{(-1)^j \lceil j/2 \rceil}(x) < x+j$, then by the inductive hypothesis we have

$$\sigma^{(-1)^j \lceil j/2 \rceil}(x) = \sigma^{(-1)^{j_0} \lceil j_0/2 \rceil}(x)$$

for some $0 \leq j_0 \leq j-1$. However, then we have

$$\sigma^{(-1)^j \lceil j/2 \rceil - (-1)^{j_0} \lceil j_0/2 \rceil}(x) = x,$$

which is a contradiction because the order of σ is k , and

$$\begin{aligned} |(-1)^j \lceil j/2 \rceil - (-1)^{j_0} \lceil j_0/2 \rceil| &\leq \lceil j/2 \rceil + \lceil j_0/2 \rceil \\ &\leq \lceil (k-1)/2 \rceil + \lceil (k-2)/2 \rceil \\ &< k. \end{aligned}$$

Hence, we must have $\sigma^{(-1)^j \lceil j/2 \rceil}(x) = x + j$, which completes the induction and the proof if $\sigma^{-1}(x) = x + 1$.

If instead $\sigma(x) = x + 1$, apply the above argument to σ^{-1} . Then $\sigma^{-1} = \sigma_X$, where $X = \{x, x + 1, \dots, x + k - 1\}$, so $\sigma = \sigma_X^{-1}$, as desired. \square

Proof of Proposition 5.3. We will use the following lemma.

Lemma. *Let G be a tournament, and let v_1, \dots, v_n be an ordering of $V(G)$ (not necessarily a matching ordering). Suppose that $v_{\pi_1(1)}, \dots, v_{\pi_1(n)}$ and $v_{\pi_2(1)}, \dots, v_{\pi_2(n)}$ are both matching orderings of G . Then G is either transitive or isomorphic to P_n , and v_1, \dots, v_n is the ordering $1 \cdot G$, where 1 is the identity element of S_n .*

Proof. We first introduce some notation: for distinct vertices u, v , let $e(u, v)$ denote the edge of G with both ends in $\{u, v\}$. For an integer i , let $\pi_i = \pi_1$ if i is odd and $\pi_i = \pi_2$ if i is even.

Now, note that an edge $e(u, v)$ is a backedge of $v_{\pi_i(1)}, \dots, v_{\pi_i(n)}$ if and only if either

- $e(u, v)$ is a backedge of v_1, \dots, v_n and $\{u, v\} \neq \{j, j + 1\}$ for any $j \equiv i \pmod{2}$,
or
- $e(u, v)$ is not a backedge of v_1, \dots, v_n and $\{u, v\} = \{j, j + 1\}$ for some $j \equiv i \pmod{2}$.

We refer to this fact as (*).

We first prove that v_1, \dots, v_n has no backedge $v_j v_i$ with $j - i \geq 2$. Suppose that v_1, \dots, v_n has a backedge $v_j v_i$ with $j - i \geq 2$. By the first point of (*), $v_j v_i$ is a backedge of both $v_{\pi_1(1)}, \dots, v_{\pi_1(n)}$ and $v_{\pi_2(1)}, \dots, v_{\pi_2(n)}$. Thus, since these two orderings are matching orderings and one of the ends of $v_j v_i$ is v_i , we have that $e(v_i, v_{i+1})$ is not a backedge of either of these orderings. However, (*) implies that any edge of the form $e(v_{i'}, v_{i'+1})$ is a backedge of exactly one of $v_{\pi_1(1)}, \dots, v_{\pi_1(n)}$ and $v_{\pi_2(1)}, \dots, v_{\pi_2(n)}$. This is a contradiction, so v_1, \dots, v_n has no backedges $v_j v_i$ with $j - i \geq 2$.

Thus, all backedges of v_1, \dots, v_n are of the form $e(v_i, v_{i+1})$ for some $1 \leq i \leq n - 1$. We claim that either no edge of this form is a backedge, or all edges of this form are backedges. Suppose the contrary. Then there is some $1 \leq i \leq n - 2$ such that exactly one of $e(v_i, v_{i+1})$, $e(v_{i+1}, v_{i+2})$ is a backedge. Let $j \in \{i, i + 1\}$ such that $e(v_j, v_{j+1})$ is not a backedge. Then by (*), $e(v_i, v_{i+1})$ and $e(v_{i+1}, v_{i+2})$ are both backedges in $v_{\pi_j(1)}, \dots, v_{\pi_j(n)}$. This is a contradiction since $v_{\pi_j(1)}, \dots, v_{\pi_j(n)}$ is a matching ordering and both these edges have end v_{i+1} , proving the claim.

Thus, v_1, \dots, v_n has no backedges $v_j v_i$ with $j - i \geq 2$, and either $e(v_i, v_{i+1})$ is not a backedge for all i , or $e(v_i, v_{i+1})$ is a backedge for all i . In the former case, G is transitive and v_1, \dots, v_n is the ordering $1 \cdot G$. In the latter case, G is P_n and v_1, \dots, v_n is the ordering $1 \cdot G$. This proves the lemma. \square

We are now ready to prove the first two points in the proposition. Suppose we have the conditions of the proposition. First, assume $\sigma = \sigma_X$. Let u_1, \dots, u_n be the ordering $v_{\pi_2(1)}, \dots, v_{\pi_2(n)}$. Then $u_{\pi_2(1)}, \dots, u_{\pi_2(n)}$ is v_1, \dots, v_n , and $u_{\pi_1(1)}, \dots, u_{\pi_1(n)}$ is $v_{\pi_2(\pi_1(1))}, \dots, v_{\pi_2(\pi_1(n))}$, which is $v_{\sigma_X(1)}, \dots, v_{\sigma_X(n)}$ since $\pi_2(\pi_1(i)) = (\pi_1 \pi_2)(i)$ and $\pi_1 \pi_2 = \sigma_X$. Thus, both $u_{\pi_1(1)}, \dots, u_{\pi_1(n)}$ and $u_{\pi_2(1)}, \dots, u_{\pi_2(n)}$ are matching orderings of G . By the Lemma, G is either transitive or isomorphic to P_n , and u_1, \dots, u_n is the ordering $1 \cdot G$. Since v_1, \dots, v_n is $u_{\pi_2(1)}, \dots, u_{\pi_2(n)}$, we have that v_1, \dots, v_n is $\pi_2 G$, as desired. The proof for $\sigma = \sigma_X^{-1}$ is analogous.

Finally, suppose $n = 4$ and $\sigma = \tau_X$. Define $e(u, v)$ as before. Call an edge a *long* edge if it has one end in $\{v_1, v_2\}$ and the other end in $\{v_3, v_4\}$. Then a long edge is a backedge of $v_{\sigma(1)}, \dots, v_{\sigma(4)}$ if and only if it is not a backedge of v_1, \dots, v_n . Thus, each long edge is a backedge of exactly one of v_1, \dots, v_4 and $v_{\sigma(1)}, \dots, v_{\sigma(4)}$. On the other hand, there are four long edges, and a matching ordering on four vertices can have at most two backedges. Thus, each of v_1, \dots, v_4 and $v_{\sigma(1)}, \dots, v_{\sigma(4)}$ have exactly two backedges, both of which are long edges. So the backedges of v_1, \dots, v_4 are either $\{v_3 v_1, v_4 v_2\}$ or $\{v_4 v_1, v_3 v_2\}$. In the first case v_1, \dots, v_n is the ordering $\pi_2 P_4$, and in the second case it is $\pi_2 \pi_1 P_4$, as desired. \square

5.5 Minimal non-matching tournaments

A tournament G is a *minimal non-matching* tournament if G is not a matching tournament and every subtournament of G with $< |V(G)|$ vertices is a matching tournament. Since every subtournament of a matching tournament is also a matching tournament, an equivalent definition of a minimal non-matching tournament is a tournament G which is not a matching tournament and for which every subtournament of G with $|V(G)| - 1$ vertices is a matching tournament.

Clearly, a tournament is not a matching tournament if and only if it has a minimal non-matching subtournament. It is then natural to ask whether the list of minimal non-matching tournaments is finite. In this section we answer in the negative.

Theorem 5.8. *There are infinitely many minimal non-matching tournaments.*

Proof. For $n \geq 3$, define the tournament Q_n as follows. Let the vertices of Q_n be v_1, \dots, v_n , and let the backedges of the ordering v_1, \dots, v_n be exactly $v_{n-1} v_1$, $v_n v_{n-2}$, and $v_{i+1} v_i$ for $1 \leq i \leq n - 3$. Notice that the tournament induced on v_1, \dots, v_{n-2} is P_{n-2} .

We will prove that for all odd $n \geq 7$, Q_n is a minimal non-matching tournament, which suffices to prove the theorem. We first show that Q_n is not a matching tournament. Suppose Q_n has a matching ordering u_1, \dots, u_n . By Proposition 5.1, u_n has indegree $n - 2$ or $n - 1$. Thus, u_n must be v_n . Let $Q' = Q_n - v_n$; then u_1, \dots, u_{n-1}

is a matching ordering of Q' . By Proposition 5.1, u_{n-1} has indegree $n-3$ or $n-2$ in Q' ; thus, u_{n-1} must be v_{n-1} . Let $Q'' = Q' - v_{n-1}$, so u_1, \dots, u_{n-2} is a matching ordering of Q'' . Now, since Q'' is isomorphic to P_{n-2} and $n \geq 7$, we have by Corollary 5.6 that u_1, \dots, u_{n-2} is either $\pi_1 Q''$ or $\pi_2 Q''$. However, for odd n , the ordering $\pi_1 Q''$ has the backedge $v_{n-2}v_{n-3}$, and the ordering $\pi_2 Q''$ has the backedge v_2v_1 . (See the two bulleted points in the proof of Proposition 5.3.) Thus, since u_1, \dots, u_n has the backedges $v_{n-1}v_1$ and v_nv_{n-2} , one of v_1, v_{n-2} is an end of two backedges of u_1, \dots, u_n , contradicting the fact that u_1, \dots, u_n is a matching ordering. So Q_n is not a matching tournament.

Now, let v be any vertex of Q_n . We wish to prove that $Q_n - v$ is a matching tournament. If $v = v_n$, consider the ordering $u_1, \dots, u_{n-2}, v_{n-1}$ of $Q_n - v_n$, where u_1, \dots, u_{n-2} is the $\pi_1 P_{n-2}$ ordering of v_1, \dots, v_{n-2} . In the $\pi_1 P_{n-2}$ ordering of v_1, \dots, v_{n-2} for odd n , v_1 is not the end of any backedge. Putting v_{n-1} after u_1, \dots, u_{n-2} adds only the backedge $v_{n-2}v_1$, so $u_1, \dots, u_{n-2}, v_{n-1}$ is a matching ordering, as desired. Similarly, if $v = v_{n-1}$, consider the ordering u_1, \dots, u_{n-1}, v_n of $Q_n - v_{n-1}$, where u_1, \dots, u_{n-2} is the $\pi_2 P_{n-2}$ ordering of v_1, \dots, v_{n-2} . In the $\pi_2 P_{n-2}$ ordering of v_1, \dots, v_{n-2} for odd n , v_{n-2} is not the end of any backedge, and adding v_n afterwards adds only the backedge v_nv_{n-2} . So u_1, \dots, u_{n-1}, v_n is a matching ordering, as desired.

Now suppose $v = v_m$ for $1 \leq m \leq n-2$. Let H_1 be the subtournament of Q_n induced on $\{v_1, \dots, v_{m-1}\}$ and let H_2 be the subtournament of Q_n induced on $\{v_{m+1}, \dots, v_{n-2}\}$. If one of these two sets is empty, then in the following argument we ignore all mentions to the corresponding subtournament. Now, H_1 and H_2 are isomorphic to P_{m-1} and P_{n-m-2} , respectively. Let u_1, \dots, u_{m-1} be the ordering $\pi_1 H_1$, and let u_{m+1}, \dots, u_{n-2} be the ordering $\pi_2 H_2$ if $n-m-2$ is odd and $\pi_1 H_2$ if $n-m-2$ is even. Consider the ordering

$$u_1, \dots, u_{m-1}, u_{m+1}, \dots, u_{n-2}, v_{n-1}, v_n$$

of $Q_n - v_m$. The ordering $u_1, \dots, u_{m-1}, u_{m+1}, \dots, u_{n-2}$ is a matching ordering of $Q_n - \{v_m, v_{n-1}, v_n\}$, and furthermore, we can check that each of v_1, v_{n-2} is either absent from this ordering or is not the end of a backedge of this ordering. Adding v_{n-1}, v_n to the end of this ordering adds at most the backedges $v_{n-1}v_1$ and $v_{n-2}v_n$, so $u_1, \dots, u_{m-1}, u_{m+1}, \dots, u_{n-2}, v_{n-1}, v_n$ is a matching ordering of $Q_n - v_m$. This completes the proof. \square

6 Excluding almost-transitive subtournaments

6.1 Almost-transitive tournaments

The set of tournaments which do not have I_n as a subtournament is finite; indeed, it is well-known that any tournament with m vertices has a transitive subtournament with $\geq \log_2 m$ vertices. We can then ask what happens when we exclude subtournaments that are *almost* transitive tournaments; that is, tournaments obtained from transitive

tournaments by reversing a small number of edges. This final section deals with the structure of tournaments that exclude such almost-transitive tournaments.

Before proceeding, we introduce some final conventions. Given a tournament G with an ordering v_1, \dots, v_n of its vertices, the *length* of a backedge $v_j v_i$ is $j - i$. If G is a tournament and u_1, \dots, u_m is an ordering of a subset of $V(G)$, we call u_1, \dots, u_m a *subordering* of G ; in addition, we will often use u_1, \dots, u_m to mean the subtournament of G induced on $\{u_1, \dots, u_m\}$. Finally, if X is a set and X° is an ordering x_1, \dots, x_n of X , and y_1, y_2 are elements not in X , we write y_1, X°, y_2 to denote the ordering $y_1, x_1, \dots, x_n, y_2$.

Now, let $n \geq 3$. Let J_n be the tournament with vertices v_1, \dots, v_n such that the ordering v_1, \dots, v_n has only one backedge $v_n v_1$. Let K_n be the tournament with vertices v_1, \dots, v_n such that this ordering has only one backedge $v_n v_2$, and let K_n^* be the tournament with vertices v_1, \dots, v_n such that this ordering has only one backedge $v_{n-1} v_1$. Note that J_n , K_n , and K_n^* have subtournaments isomorphic to J_m , K_m , and K_m^* , respectively, for $n \geq m$.

We also have the following alternate definitions: Let C_3 be the cyclic triangle, let D_4 to be the tournament with vertices ordered as v_1, v_2, v_3, v_4 such that $v_1 \Rightarrow \{v_2, v_3, v_4\}$ and $\{v_2, v_3, v_4\}$ forms a cyclic triangle, and let D_4^* be the tournament with vertices ordered as v_1, v_2, v_3, v_4 such that $\{v_2, v_3, v_4\} \Rightarrow v_1$ and $\{v_2, v_3, v_4\}$ forms a cyclic triangle. Then J_n is $C_3(I_1, I_1, I_{n-2})$, K_n is $D_4(I_1, I_1, I_{n-3})$, and K_n^* is $D_4^*(I_1, I_1, I_{n-3})$.

In the first subsection, we state and prove a theorem by P. Seymour on the structure of tournaments that exclude J_n subtournaments. In the second subsection, we prove our main result, which is a structure theorem for tournaments that exclude both K_n and K_n^* .

6.2 Excluding J_n

Seymour's theorem is as follows.

Theorem 6.1 (Seymour). *The following hold.*

- (i) *For each $n \geq 3$, there exists k such that if a tournament G has no subtournament isomorphic to J_n , then $V(G)$ has an ordering $v_1, \dots, v_{|V(G)|}$ that has no backedge of length $> k$.*
- (ii) *For each $k \geq 1$, there exists n such that if a tournament G has an ordering $v_1, \dots, v_{|V(G)|}$ of its vertices with no backedge of length $> k$, then G has no subtournament isomorphic to J_n .*

Proof of (i). We will prove that (i) holds with $k = 10n \cdot 2^{2n}$. Let G be a tournament that has no subtournament isomorphic to J_n . Let I be a transitive subtournament of G which maximizes $V(I)$, and let $u_1, \dots, u_{V(I)}$ be the vertices of I so that $u_i \rightarrow u_j$ if $i < j$. Suppose $v \in V(G) \setminus V(I)$. Let $A = A_G(v) \cap V(I)$ and $B = B_G(v) \cap V(I)$. Since G has no J_n subtournament, there are no integers $1 \leq i_1 < \dots < i_{n-1} \leq |V(I)|$

such that the subordering $u_{i_1}, \dots, u_{i_{n-1}}, v$ has only one backedge vu_{i_1} , or such that the subordering $v, u_{i_1}, \dots, u_{i_{n-1}}$ has only one backedge $u_{i_{n-1}}v$. Thus, there are no integers $1 \leq i_1 < \dots < i_{n-1} \leq |V(I)|$ such that either

- (1) $u_{i_1} \in A$ and $u_{i_2}, \dots, u_{i_{n-1}} \in B$, or
- (2) $u_{i_1}, \dots, u_{i_{n-2}} \in A$ and $u_{i_n} \in B$.

Now, if either A or B is empty, then $I + v$ is transitive, contradicting the maximality of I . So both A and B are nonempty. Let $a_v = \min\{i : u_i \in A\}$ and $b_v = \max\{i : u_i \in B\}$. If $a_v > b_v$, then since $u_i \in B$ for all $i < a_v$ and $u_i \in A$ for all $i > b_v$, we must have $a_v - b_v = 1$. But then $u_1, \dots, u_{b_v}, v, u_{a_v}, \dots, u_{|V(I)|}$ is a transitive subtournament of G , contradicting the maximality of I . So $a_v < b_v$.

We now claim that $b_v - a_v < 2n$. To see this, let $Y = \{u_i : a_v < i < b_v\}$. Since $a_v \in A$, by fact (1) from above we have that $|Y \cap B| < n - 1$. Similarly, since $b_v \in B$, by (2) we have that $|Y \cap A| < n - 1$. Hence, $|Y| < 2n - 2$, so $b_v - a_v < 2n - 1 < 2n$, as claimed.

Thus, to each vertex $v \in V(G) \setminus V(I)$, we have associated integers $a_v < b_v$ such that $u_i \rightarrow v$ for all $i < a_v$, $v \rightarrow u_i$ for all $i > b_v$, and $b_v - a_v < 2n$. In particular, $v \rightarrow u_i$ for all $i \geq a_v + 2n$. For each $1 \leq a \leq |I(V)| - 1$, let X_a denote the set of vertices $v \in V(G) \setminus V(I)$ such that $a_v = a$. Then the sets X_a form a partition of $V(G) \setminus V(I)$, and for all a ,

$$\{u_1, \dots, u_{a-1}\} \Rightarrow \{u_a\} \cup X_a \Rightarrow \{u_{a+2n}, \dots, u_{|V(I)|}\}. \quad (\dagger)$$

Now, we claim that $|X_a| \leq 2^{2n}$ for all a . Indeed, suppose $|X_a| \geq 2^{2n} + 1$ for some a . Then the subtournament of G induced on X_a has a transitive subtournament with $\lceil \log_2(2^{2n} + 1) \rceil = 2n + 1$ vertices. Let $X \subseteq X_a$ be the vertices of this transitive subtournament. By (\dagger) , the subtournament of G induced on

$$\{u_1, \dots, u_{a-1}\} \cup X \cup \{u_{a+2n}, \dots, u_{|V(I)|}\}$$

is transitive. However, since $|X| = 2n + 1$, this transitive tournament has $|V(I)| + 1$ vertices, contradicting the maximality of I . So $|X_a| \leq 2^{2n}$ for all a , as claimed.

Now, for each a , let X_a° be an arbitrary ordering of X_a . (The ordering is empty if X_a is empty.) We claim that the ordering

$$u_1, X_1^\circ, u_2, X_2^\circ, u_3, X_3^\circ, \dots, u_{|V(I)|-1}, X_{|V(I)|-1}^\circ, u_{|V(I)|} \quad (*)$$

of $V(G)$ has no backedge of length $> 10n \cdot 2^{2n}$. This will suffice to prove (i).

Suppose $v'v$ is a backedge of $(*)$. Let a be such that $v \in \{u_a\} \cup X_a$, and let j be such that $v' \in \{u_j\} \cup X_j$. First suppose that $j \leq a + 3n - 1$. Then the set of vertices between v and v' in $(*)$ are a subset of $X_a \cup \{u_{a+1}\} \cup X_{a+1} \cup \dots \cup \{u_j\} \cup X_j$. Since $j \leq a + 3n - 1$, and $|X_i| \leq 2^{2n}$ for all i , there are thus at most $3n \cdot 2^{2n} + 3n - 1$ vertices between v and v' in $(*)$. Thus, the length of $v'v$ is at most $3n \cdot 2^{2n} + 3n < 10n \cdot 2^{2n}$, as desired.

Now suppose $j \geq a + 3n$. By (\dagger) , we have $v \Rightarrow \{u_{a+2n}, u_{a+2n+1}, \dots, u_{a+3n-3}\} \Rightarrow v'$. By assumption, we have $v' \rightarrow v$. Thus, $v, u_{a+2n}, \dots, u_{a+3n-3}, v'$ is isomorphic to J_n , a contradiction. So we cannot have such a backedge $v'v$. This proves (i). \square

Proof of (ii). We prove that (ii) holds with $n = 10k$. Let G be a tournament that has an ordering $v_1, \dots, v_{|V(G)|}$ of $V(G)$ with no backedges of length $> k$. Suppose that G has a subtournament isomorphic to J_{10k} . Let u_1, \dots, u_{10k} be an ordering of the vertices of this subtournament such that $u_{10k}u_1$ is the only backedge. Now, let i, j be such that $u_1 = v_i$ and $u_{10k} = v_j$. First suppose $i < j$. Take any vertex $u \in \{u_2, \dots, u_{10k-1}\}$, and let x be such that $u = v_x$. We consider the number of possible values of x . Since $i < j$, the edge $v_jv_i = u_{10k}u_1$ is a backedge of $v_1, \dots, v_{|V(G)|}$, and hence has length $\leq k$. So if $i < x < j$ then there are $< k$ possible values of x . If $x < i$, then $v_iv_x = u_1u$ is a backedge of $v_1, \dots, v_{|V(G)|}$ and hence has length $\leq k$, so there $\leq k$ possible values of x . Similarly if $x > j$, then $v_xv_j = uu_{10k}$ is a backedge of $v_1, \dots, v_{|V(G)|}$ and hence has length $\leq k$, so there are $\leq k$ possible values of x . In total there are $\leq 3k$ possible values of x . This holds for all $u \in \{u_2, \dots, u_{10k-1}\}$, a contradiction since this set has $10k - 2 > 3k$ vertices.

Now suppose $j < i$. Again, take a vertex $u \in \{u_2, \dots, u_{10k-1}\}$, let $u = v_x$, and consider the number of possible values for x . If $x < i$, then $v_iv_x = u_1u$ is a backedge of $v_1, \dots, v_{|V(G)|}$ and hence has length $\leq k$, so there are $\leq k$ possible values for x . If $x > i$, then $x > j$, so $v_xv_j = uu_{10k}$ is a backedge of $v_1, \dots, v_{|V(G)|}$, and as before there are $\leq k$ possible values of x . In total there are $\leq 2k$ possible values of x , which as before is a contradiction. This completes the proof. \square

6.3 Excluding K_n and K_n^*

The main theorem presented in this section is the following.

Theorem 6.2. *The following hold.*

- (i) *For each $n \geq 3$, there exists k such that if a tournament G has no subtournament isomorphic to K_n or K_n^* , then G can be written as $T_r(H_1, \dots, H_r)$, where $r \geq 1$ is odd and for each $1 \leq i \leq r$, the tournament H_i has no subtournament isomorphic to J_k .*
- (ii) *For each $k \geq 3$, there exists n such that if a tournament G can be written as $T_r(H_1, \dots, H_r)$, where $r \geq 1$ is odd and for each $1 \leq i \leq r$, the tournament H_i has no subtournament isomorphic to J_k , then G has no subtournament isomorphic to K_n or K_n^* .*

Proof of (i). For each $n \geq 3$, we will prove that there exists a large enough integer m such that (i) holds with $k = \max(10 \cdot 2^m, 100n \cdot 2^{2n})$. Fix $m > 0$ for now; we will increase m later if needed. Let G be a tournament with no subtournament isomorphic to K_n or K_n^* . If $|V(G)| < 2^m$, then there is certainly an ordering of $V(G)$ with no backedge of length $> 2^m$; thus, by Theorem 6.1(ii) (more specifically, the proof of

Theorem 6.1(ii)), G has no subtournament isomorphic to $J_{10 \cdot 2^m}$. We can thus write G as $T_1(G)$ where G has no subtournament isomorphic to $J_{\max(10 \cdot 2^m, 100n \cdot 2^{2n})}$, as desired.

So assume $|V(G)| \geq 2^m$. Let I be a transitive subtournament of G which maximizes $V(I)$, and let $u_1, \dots, u_{|V(I)|}$ be the vertices of I so that $u_i \rightarrow u_j$ if $i < j$. Since $|V(G)| \geq 2^m$, we have $|V(I)| \geq m$. Now, let $X \subseteq V(G) \setminus V(I)$ be the set of vertices $v \in V(G) \setminus V(I)$ such that either $u_1 \rightarrow v$ or $v \rightarrow u_{|V(I)|}$. Let $N = V(G) \setminus (V(I) \cup X)$.

Our proof consists of two main steps. First, we use an argument similar to the one in the previous proof to show that there is an ordering of $V(I) \cup X$ in which all backedges are of bounded length. Afterwards, we show that $V(I) \cup X$ and N can be broken up into homogeneous sets that can be “weaved” together to give the desired form.

Step 1. Let $v \in X$, and let $A = A_G(v) \cap V(I)$ and $B = B_G(v) \cap V(I)$. By the definition of X , either $u_1 \in B$ or $u_{|V(I)|} \in A$. Now, since G does not have a K_n subtournament, there are no integers $1 \leq i_1 < \dots < i_{n-1} \leq |V(I)|$ such that

- the suborder $u_{i_1}, v, u_{i_2}, \dots, u_{i_{n-1}}$ has only one backedge $u_{i_{n-1}}v$, or
- the suborder $u_{i_1}, u_{i_2}, \dots, u_{i_{n-1}}, v$ has only one backedge vu_{i_2} .

Also, since G has no K_n^* subtournament, there are no integers $1 \leq i_1 < \dots < i_{n-1} \leq |V(I)|$ such that

- the suborder $v, u_{i_1}, \dots, u_{i_{n-2}}, u_{i_{n-1}}$ has only one backedge $u_{i_{n-2}}v$, or
- the suborder $u_{i_1}, \dots, u_{i_{n-2}}, v, u_{i_{n-1}}$ has only one backedge vu_{i_1} .

In total, there are no integers $1 \leq i_1 < \dots < i_{n-1} \leq |V(I)|$ such that any of the following hold.

- (1) $u_{i_1} \in B$, $u_{i_2}, \dots, u_{i_{n-2}} \in A$, and $u_{i_{n-1}} \in B$.
- (2) $u_{i_1} \in B$, $u_{i_2} \in A$, $u_{i_3}, \dots, u_{i_{n-1}} \in B$.
- (3) $u_{i_1}, \dots, u_{i_{n-3}} \in A$, $u_{i_{n-2}} \in B$, $u_{i_{n-1}} \in A$.
- (4) $u_{i_1} \in A$, $u_{i_2}, \dots, u_{i_{n-2}} \in B$, and $u_{i_{n-1}} \in A$.

Now, as in the proof of Theorem 6.1(i), A and B are both nonempty. Let $a_v = \min\{i : u_i \in A\}$ and $b_v = \max\{i : u_i \in B\}$. Again as in the previous proof, we have $a_v < b_v$. We claim that $b_v - a_v < 2n$. Let $Y = \{u_i : a_v < i < b_v\}$. Since $v \in X$, we have either $u_1 \in B$ or $u_{|V(I)|} \in A$. Suppose $u_1 \in B$. Since $u_1, u_{b_v} \in B$, by fact (1) from above we have that $|Y \cap A| < n - 2$. Also, since $u_1 \in B$ and $u_{a_v} \in A$, by fact (2) we have that $|Y \cap B| < n - 2$. Thus, $|Y| < 2n - 4$. Similarly, if $u_{|V(I)|} \in A$, by fact (3) we have $|Y \cap A| < n - 2$, and by fact (4) we have $|Y \cap B| < n - 2$, so $|Y| < 2n - 4$. In either case we have $b_v - a_v < 2n - 3 < 2n$, as claimed.

Thus, as in the previous proof, to each vertex $v \in X$ we have associated integers $a_v < b_v$ such that $u_i \rightarrow v$ for all $i < a_v$, $v \rightarrow u_i$ for all $i > b_v$, and $b_v - a_v < 2n$. For

each $1 \leq a \leq |I(V)| - 1$, let X_a denote the set of vertices $v \in X$ such that $a_v = a$. Then the sets X_a form a partition of X , and by the argument in the previous proof, $|X_a| \leq 2^{2n}$ for all n .

For each a , let X_a° be an arbitrary ordering of X_a . We claim that the ordering

$$u_1, X_1^\circ, u_2, X_2^\circ, u_3, X_3^\circ, \dots, u_{|V(I)|-1}, X_{|V(I)|-1}^\circ, u_{|V(I)|} \quad (*)$$

of $V(I) \cup X$ has no backedge of length $> 10n \cdot 2^{2n}$. Suppose $v'v$ is a backedge of $(*)$. Let a be such that $v \in \{u_a\} \cup X_a$, and let j be such that $v' \in \{u_j\} \cup X_j$. If $j \leq a + 3n - 1$, then as in the previous proof the length of $v'v$ is at most $10n \cdot 2^{2n}$, as desired. Now suppose $j \geq a + 3n$. As in the previous proof, we have $v \Rightarrow \{u_{a+2n}, u_{a+2n+1}, \dots, u_{a+3n-4}\} \Rightarrow v'$ and $v' \rightarrow v$. Moreover, we have $u_{b_v} \Rightarrow v \cup \{u_{a+2n}, \dots, u_{a+3n-4}\} \cup v'$. Thus, $u_{b_v}, v, u_{a+2n}, \dots, u_{a+3n-4}, v'$ is isomorphic to K_n , a contradiction. So we cannot have such a backedge $v'v$, proving the claim and completing this step. \square

Now, let $M = V(I) \cup X$, and let $v_1, \dots, v_{|M|}$ be the ordering $(*)$ of M . Thus, $v_1, \dots, v_{|M|}$ has no backedges of length $> 10n \cdot 2^{2n}$. So by the proof of Theorem 6.1(ii), the subtournament of G induced on M has no subtournament isomorphic to $J_{100n \cdot 2^{2n}}$. If N is empty, then $M = V(G)$, so in this case G has no subtournament isomorphic to $J_{100n \cdot 2^{2n}}$; we can then write G as $T_1(G)$, where G has no subtournament isomorphic to $J_{\max(10 \cdot 2^m, 100n \cdot 2^{2n})}$, as desired. If N is not empty, we proceed to Step 2.

Step 2. Our goal is to partition M into nonempty sets M_1, M_2, \dots, M_{p+1} and partition N into nonempty sets N_1, N_2, \dots, N_p such that none of the subtournaments induced on $M_1, \dots, M_{p+1}, N_1, \dots, N_p$ have a subtournament isomorphic to $J_{\max(10 \cdot 2^m, 100n \cdot 2^{2n})}$, and $M_i \Rightarrow M_j$ if $i < j$, $N_i \Rightarrow N_j$ if $i < j$, $N_j \Rightarrow M_i$ if $i \leq j$, and $M_j \Rightarrow N_i$ if $i < j$. If this is the case, then we can write G as $G'(M_1, N_1, M_2, N_2, \dots, M_p, N_p, M_{p+1})$, where G' is a tournament with vertices ordered as t_1, \dots, t_{2p+1} such that

- $t_i \rightarrow t_j$ if $i < j$ and i, j have the same parity.
- $t_j \rightarrow t_i$ if $i < j$ and i, j have opposite parity.

Considering the vertices of G' in the order $t_{2p+1}, t_{2p}, \dots, t_1$, we see that G' is in fact a BB weave. By Proposition 3.2, G' is thus isomorphic to T_{2p+1} . So G is of the desired form, which will complete the proof.

Since $v_1, \dots, v_{|M|}$ is the ordering $(*)$ of M , we have $v_1 = u_1$ and $v_{|M|} = u_{|V(I)|}$. Thus, by the definition of N , we have $N \Rightarrow v_1$ and $v_{|M|} \Rightarrow N$. It follows that the subtournament of G induced on N does not have a subtournament isomorphic to J_{n-1} ; if there was such a subtournament H , then $H + v_1$ would be isomorphic to K_n , a contradiction.

Now, for each $w \in N$, let $A_w = A_G(w) \cap M$ and $B_w = B_G(w) \cap M$. Then $v_1 \in A_w$ and $v_{|M|} \in B_w$ for all $w \in N$. We prove the following claim.

Claim 1. For all $w \in N$, $A_w \Rightarrow B_w$.

Proof. Let $w \in N$. Since $v_1 \in A_w$ and $v_{|M|} \in B_w$, A_w and B_w are nonempty. Let $b = \min\{i : v_i \in B_w\}$ and $a = \max\{i : v_i \in A_w\}$. We claim that either $b > 200n^2 \cdot 2^{2n}$ or $a < |M| - 200n^2 \cdot 2^{2n}$. Suppose the contrary. Then $a - b \geq |M| - 400n^2 \cdot 2^{2n}$. Since $|M| \geq |V(I)| \geq m$, we can choose m large enough so that $a - b > 200n^2 \cdot 2^{2n}$.

Now, let $Y = \{v_i : b < i < a\}$, so $|Y| \geq 200n^2 \cdot 2^{2n}$. Suppose that $|Y \cap A| \geq 100n^2 \cdot 2^{2n}$. Let $b < i_1 < i_2 < \dots < i_{100n^2 \cdot 2^{2n}} < a$ be such that $v_{i_s} \in Y \cap A$ for all $1 \leq s \leq 100n^2 \cdot 2^{2n}$. Consider the subordering

$$v_{i_{100n^2 \cdot 2^{2n}}}, v_{i_{200n^2 \cdot 2^{2n}}}, v_{i_{300n^2 \cdot 2^{2n}}}, \dots, v_{i_{100n(n-3) \cdot 2^{2n}}}.$$

This subordering of $v_1, \dots, v_{|M|}$ has no backedges, because $v_1, \dots, v_{|M|}$ has no backedge of length $> 10n \cdot 2^{2n}$. For the same reason, we have

$$v_b \Rightarrow \{v_{i_{100n^2 \cdot 2^{2n}}}, v_{i_{200n^2 \cdot 2^{2n}}}, \dots, v_{i_{100n(n-3) \cdot 2^{2n}}}\} \Rightarrow v_{|M|}$$

as well as $v_b \rightarrow v_{|M|}$. Thus, since $v_b, v_{|M|} \in B$ and $v_{i_{100n^2 \cdot 2^{2n}}}, \dots, v_{i_{100n(n-3) \cdot 2^{2n}}} \in A$, we have that

$$v_b, w, v_{i_{100n^2 \cdot 2^{2n}}}, v_{i_{200n^2 \cdot 2^{2n}}}, \dots, v_{i_{100n(n-3) \cdot 2^{2n}}}, v_{|M|}$$

is isomorphic to J_n , a contradiction. Hence, we must have $|Y \cap A| < 100n^2 \cdot 2^{2n}$.

An analagous argument using J_n^* shows that $|Y \cap B| < 100n^2 \cdot 2^{2n}$. This contradicts $|Y| \geq 200n^2 \cdot 2^{2n}$. Hence, we must have either $b > 200n^2 \cdot 2^{2n}$ or $a < |M| - 200n^2 \cdot 2^{2n}$, as claimed.

Now, assume that $b > 200n^2 \cdot 2^{2n}$. Then $v_i \in A$ for all $i \leq 200n^2 \cdot 2^{2n}$. Suppose that we do not have $A_w \Rightarrow B_w$, so there are two vertices $v_i \in A$ and $v_j \in B$ with $v_j \rightarrow v_i$. Since $v_j \in B$, we have $j > 200n^2 \cdot 2^{2n}$. Then, since $v_j \rightarrow v_i$ and $v_1, \dots, v_{|M|}$ has no backedge of length $> 10n \cdot 2^{2n}$, we have $i > 150n^2 \cdot 2^{2n}$. Thus, the subordering

$$v_{100n^2 \cdot 2^{2n}}, v_{200n^2 \cdot 2^{2n}}, v_{300n^2 \cdot 2^{2n}}, \dots, v_{100n(n-3) \cdot 2^{2n}}, v_j, v_i$$

has no backedge, since $v_j \rightarrow v_i$ and $v_1, \dots, v_{|M|}$ has no backedge of length $> 10n \cdot 2^{2n}$. But $v_{100n^2 \cdot 2^{2n}}, \dots, v_{100n(n-3) \cdot 2^{2n}} \in A$ (since $100n(n-3) \cdot 2^{2n} < 200n^2 \cdot 2^{2n}$), and $v_j \in B$, $v_i \in A$ by definition, so we have that

$$w, v_{100n^2 \cdot 2^{2n}}, v_{200n^2 \cdot 2^{2n}}, v_{300n^2 \cdot 2^{2n}}, \dots, v_{100n(n-3) \cdot 2^{2n}}, v_j, v_i$$

is isomorphic to J_n^* . This is a contradiction. Hence, if $b > 200n^2 \cdot 2^{2n}$, then $A_w \Rightarrow B_w$. By an analagous argument using J_n , if $a < |M| - 200n^2 \cdot 2^{2n}$, then $A_w \Rightarrow B_w$. Either way, $A_w \Rightarrow B_w$, as desired. \square

Now, the subtournament of G induced on M can be written as $I_s(S_1, \dots, S_s)$, where S_1, \dots, S_m are the strong components of G . Since $A_w \Rightarrow B_w$ for each $w \in N$, we have for each $w \in N$ that $A_w = V(S_1) \cup V(S_2) \cup \dots \cup V(S_{s_w})$ and $B_w = V(S_{s_w+1}) \cup \dots \cup V(S_s)$ for some s_w . It follows that for every $w, w' \in N$, either $A_w \subseteq A_{w'}$ or $A_{w'} \subseteq A_w$; in other words, the \subseteq relation on $\{A_w\}_{w \in N}$ gives a total order on N . By defining an equivalence relation \sim on N so that $w \sim w'$ if and only if $A_w = A_{w'}$, we

can thus partition N into equivalence classes so that the total order on N becomes a strict total order on the equivalence classes; in other words, we can partition N into nonempty sets N_1, N_2, \dots, N_p such that $A_w = A_{w'}$ for every $w, w' \in N_i$, and $A_w \subsetneq A_{w'}$ if $w \in N_i, w' \in N_j$ for $i < j$.

For each $1 \leq i \leq p$, define A_i to equal A_w for any $w \in N_i$. Thus, $A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_p$. Now, let $M_1 = A_1$, let $M_i = A_i \setminus A_{i-1}$ for $2 \leq i \leq p$, and let $M_{p+1} = M \setminus A_p$. Then M_i is nonempty for $2 \leq i \leq p$, and M_1, M_{p+1} are nonempty because $v_1 \in M_1$ and $v_{|M|} \in M_{p+1}$ (since $v_1 \in A_w$ and $v_{|M|} \notin A_w$ for all $w \in N$). Thus, M_1, \dots, M_{p+1} is a partition of M .

In summary, we have a partition M_1, \dots, M_{p+1} of M and a partition N_1, \dots, N_p of N so that for all i , $N_i \Rightarrow M_1 \cup M_2 \cup \dots \cup M_i$ and $M_{i+1} \cup \dots \cup M_{p+1} \Rightarrow N_i$. By Claim 1, $A_i \Rightarrow M \setminus A_i$ for all i , from which it follows that $M_i \Rightarrow M_j$ if $i < j$. We also have the following.

Claim 2. *If $i < j$, then $N_i \Rightarrow N_j$.*

Proof. Let $1 \leq i < j \leq p$, and suppose there are vertices $w_i \in N_i, w_j \in N_j$ such that $w_j \rightarrow w_i$. Then we have the following relations.

- (a) $w_j \Rightarrow \{w_i\} \cup M_1 \cup \dots \cup M_j$
- (b) $w_i \Rightarrow M_1 \cup \dots \cup M_i \Rightarrow M_{i+1} \cup \dots \cup M_j \Rightarrow w_i$
- (c) $\{w_j\} \cup M_{i+1} \cup \dots \cup M_{p+1} \Rightarrow w_i$
- (d) $w_j \Rightarrow M_{i+1} \cup \dots \cup M_j \Rightarrow M_{j+1} \cup \dots \cup M_{p+1} \Rightarrow w_j$

Relations (a) and (b) imply that G contains the subtournament

$$D_4(w_j, w_i, M_1 \cup \dots \cup M_i, M_{i+1} \cup \dots \cup M_j)$$

where we are abusing notation and using sets of vertices to mean the subtournaments of G induced on them. (Note that the above expression is valid because all four sets used as arguments to $D_4(\cdot, \cdot, \cdot, \cdot)$ are nonempty.) It follows that if either $M_1 \cup \dots \cup M_i$ or $M_{i+1} \cup \dots \cup M_j$ has a transitive subtournament with $n - 3$ vertices, then G has a subtournament isomorphic to $D_4(I_1, I_1, I_1, I_{n-3}) = K_n$, a contradiction. Similarly, (c) and (d) imply that G contains the subtournament

$$D_4^*(w_i, w_j, M_{i+1} \cup \dots \cup M_j, M_{j+1} \cup \dots \cup M_{p+1}).$$

($M_{j+1} \cup \dots \cup M_{p+1}$ is nonempty because $j \leq p$.) If either $M_{i+1} \cup \dots \cup M_j$ or $M_{j+1} \cup \dots \cup M_{p+1}$ has a transitive subtournament with $n - 3$ vertices, then G has a subtournament isomorphic to $D_4^*(I_1, I_1, I_1, I_{n-3}) = K_n^*$, a contradiction.

Thus, none of the subtournaments induced on $M_1 \cup \dots \cup M_i, M_{i+1} \cup \dots \cup M_j$, and $M_{j+1} \cup \dots \cup M_{p+1}$ has a transitive subtournament with $n - 3$ vertices. But these three sets form a partition of M , and since $|M| \geq m$, for large enough m at least one of these sets has size $\geq 2^n$. The subtournament induced on this set has a transitive subtournament with $n - 3$ vertices, a contradiction. So no such w_i, w_j exist, and hence $N_i \Rightarrow N_j$, as desired. \square

Now, we have partitioned $V(G)$ into nonempty sets $M_1, \dots, M_{p+1}, N_1, \dots, N_p$ such that $M_i \Rightarrow M_j$ if $i < j$, $N_i \Rightarrow N_j$ if $i < j$, $N_j \Rightarrow M_i$ if $i \leq j$, and $M_j \Rightarrow N_i$ if $i < j$. Furthermore, as we noted earlier, the subtournament of G induced on M has no subtournament isomorphic to $J_{100n \cdot 2^{2n}}$, and the subtournament of G induced on N has no subtournament isomorphic to J_{n-1} . Hence, none of the subtournaments of G induced on $M_1, \dots, M_{p+1}, N_1, \dots, N_p$ have a subtournament isomorphic to $J_{\max(10 \cdot 2^m, 100n \cdot 2^{2n})}$. We have thus achieved our goal, and the proof of (i) is complete. \square

Proof of (ii). We prove that (ii) holds with $n = k + 1$. Let G be a tournament that can be written as $T_r(H_1, \dots, H_r)$, where $r \geq 1$ is odd and for each $1 \leq i \leq r$, H_i has no subtournament isomorphic to J_k . If $r = 1$, then $G = H_1$, so G has no subtournament isomorphic to J_k ; thus, G certainly has no subtournament isomorphic to K_{k+1} or K_{k+1}^* , since J_k is a subtournament of these tournaments. So assume $r \geq 3$. Let $r = 2\ell + 1$. If G has a subtournament isomorphic to K_{k+1} , there must be some vertex v of G whose outneighborhood contains a subtournament isomorphic to J_k . However, the outneighborhood of every $v \in V(G)$ is of the form $I_\ell(H_i, H_{i+1}, \dots, H_{i+\ell-1})$, where the indices of the H subtournaments are taken modulo r . Since J_k is strongly connected and each H_j does not have a J_k subtournament, $I_\ell(H_i, H_{i+1}, \dots, H_{i+\ell-1})$ does not have one either. Thus G does not contain K_{k+1} . By an analogous argument, G does not have a subtournament isomorphic to K_{k+1}^* , completing the proof. \square

7 Further questions

To conclude, we list a few directions that future research might take. First, it would be nice to find a good application for Theorem 3.1. As noted in the Introduction, Schmerl and Trotter's Theorem 2.11 has often been used in the study prime tournaments. Since Theorem 3.1 appears to be a substantive improvement on this theorem, it may help in proving further facts about the prime tournaments. Our proof of Theorem 3.8 gives an example of how the theorem might be used.

In Section 5, we looked at tournaments whose vertices can be ordered so that the backedges form a matching. One can generalize this and look at tournaments with orderings in which the backedges form different structures; for example, forests. It would be good to know bounds on the number of such orderings that a prime tournament can have, and whether there are infinitely many minimal tournaments which do not have such orderings.

Finally, there is much work to be done on looking at families of tournaments which exclude certain subtournaments. Chudnovsky and Seymour very recently proved a structure theorem for the tournaments that exclude both of the following two subtournaments: for odd $n \geq 3$, $n = 2k + 1$, let L_n be the tournament with vertices v_1, \dots, v_n so that this ordering has only one backedge $v_n v_{k+1}$, and let L_n^* be the tournament with vertices v_1, \dots, v_n so that this ordering has only one backedge $v_{k+1} v_1$.

Their result centers around tournaments that are obtained from $T_r(I^1, \dots, I^r)$, where $r \geq 1$ is odd and I^1, \dots, I^r are transitive tournaments, by reversing edges which are in a sense “extreme.” Another, somewhat different result by Latka [6] gives the structure of tournaments which exclude W_5 ; in particular, the prime tournaments which do not have W_5 subtournaments are precisely I_2 , T_n and U_n for odd $n \geq 1$, the Paley tournament with 7 vertices, and the tournament obtained from the Paley tournament with 7 vertices by deleting a vertex.

It would be nice to have a better understanding of the tournaments which exclude T_5 and U_5 subtournaments. We also want to consider tournaments which exclude the following tournament: for $n = 3k + 2$, $k \geq 1$, let E_n be the tournament with vertices v_1, \dots, v_n so that this ordering has only one backedge $v_{2k+2}v_{k+1}$. Part of the difficulty of this latter problem comes from our lack of understanding of even the case E_5 . A structural theorem for the tournaments that exclude E_5 would not only aid in this problem, but also be interesting in itself.

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This thesis represents my own work in accordance with University regulations.